Kathy Lomax  
Office of Sponsored Programs  
University of Wisconsin – Platteville

Hi Kathy,

I would like to thank you, scholarly activity improvement fund committee and Prof. Carol Sue Butts for providing me SAIF A grant to work on my scholarly activity. My report starts with related publications, then, the abstract of my project and ends with results along with conclusion.


Title: Momentum evolution of ejected and entrained fluid regions during laminar vortex ring formation for application to mixing

Type of project: Research

Abstract

Ambient fluid entrainment from a piston-cylinder mechanism has been studied, and it has been observed that the ejected fluid is dominant in the forming vortex ring during the piston motion, with the majority of entrainment taking place after the piston stops. The present interest will be in the momentum evolution of entrained and ejected fluid, and momentum exchange among the ejected, entrained fluid and added mass. To this end, vortex rings will be generated numerically by transient jet ejection for fluid slug length (length – to – diameter \((L/D)\)) ratios of 0.5 – 3.0, three different velocity programs (trapezoidal, triangular negative slope (NS) and positive slope (PS)) and jet Reynolds number of 1000. Lagrangian coherent structures (LCS) will be utilized to identify ejected and entrained fluid boundaries, and a Runge-Kutta 4\(^{th}\) order scheme will be used for advecting these boundaries with numerical velocity data. By monitoring the center of mass of these fluid boundaries, momentum of each component will be calculated individually and related to the total impulse provided by the vortex ring generator.
Publications

- Results of 2008 SAIF A grant work have been presented in the American Physical Society – Division of Fluid Dynamics conference in November 2008. Also, this work has been published in the Bulletin of the American Physical Society, and a paper has been sent to Journal of Theoretical and Computational Fluid Dynamics and this paper is currently under review process.


Here is the result and conclusion parts of the paper I am working on:

Results

A. Time evolution of ejected and entrained fluid

Figure 7 illustrates the evolution of ejected and entrained fluid for \( L/D = 2.0 \) and the trapezoidal velocity program. The ejected fluid’s shape initially looks like a cylinder as one would expect; however, the back edge is not straight like the front edge. This actually indicates that fluid near \( r = D/2 \) is not pulled into the ring toward the end of the jet pulse. As the flow evolves, the entrained fluid domain is swept over the developing vortex ring spiral by Biot-Savart induction and enters the spiral through the rear such that the union of the two fluid
domains gives the expected ellipsoidal shape to the vortex ring boundary in figure 7(e) and figure 8. The initial location and time evolution of the ejected and entrained fluid domains agrees qualitatively with Peng and Dabiri [27] although they do not investigate the two domains separately.

The precise overlapping of the domain boundaries illustrated in figure 8 confirms the accuracy of the advection scheme. As further confirmation of the results, comparing the final volume of the vortex ring with the experimental results of Olcay and Krueger [26] (for $Re_j = 1000, L/D = 2.0$ and trapezoidal velocity program) gives less than 5 % difference, which is within experimental uncertainty.
Fig. 7 Evolution of ejected and entrained fluid. Panels (a), (b), (c), (d), and (e) show the ejected and entrained fluid at $t^* = 0.0, 0.5, 1.0, 1.5, \text{ and } 2.0, \text{ respectively for the trapezoidal velocity program with } L/D = 2.0$. The red solid line and blue dash-dot line represent ejected fluid and entrained fluid domains, respectively.
Fig. 8 Enlarged view of ejected and entrained fluid regions at $t^* = 2.0$ for the trapezoidal velocity program with $L/D = 2.0$.

**B. The effect of velocity program and $L/D$ on the initial location of the entrained fluid boundary**

As illustrated in the preceding section, LCS at $t^* = 0$ identifies the location of ambient fluid that will eventually be entrained into the vortex ring. Such information is especially useful for applications involving mixing since it indicates where fluid components should reside at $t^* = 0$ to mix most effectively. While the present results are limited to laminar flow ($Re_J = 1000$) with a highly organized "mixing" process (see figure 7), the entrained fluid domains identified herein may also provide insight into applications requiring fine-scale mixing (such as chemical reactions) since the vortex rings can break down and become turbulent downstream for slightly higher Reynolds number. Also, the initial shear layer roll-up for rings that are turbulent immediately following formation is qualitatively similar to laminar vortex ring roll-up (see figure 8 in Glezer [12]).

To analyze the effect of jet parameters on the "mixing" process, LCSs are used to determine the size and shape of the entrained fluid region at $t^* = 0$ for different $L/D$ and velocity programs. Figure 9 shows the initial shape of the entrained fluid regions for the trapezoidal,
triangular NS and PS velocity programs with $L/D = 2.0$. As expected from the results of Olcay and Krueger [26], the domain sizes are larger for the trapezoidal and triangular NS velocity programs. Likewise, the trapezoidal and triangular NS velocity programs entrain a larger fraction of ambient fluid from in front of the nozzle ($x > 0$), which is a consequence of the stronger overall Biot-Savart induction provided by the strong initial shear layer associated with the rapid jet initiation.

Fig. 9 Shape of the entrained fluid regions at $t^* = 0$ for the trapezoidal, triangular NS and PS velocity programs with $L/D = 2.0$.

Figure 10 illustrates the initial shape of the entrained fluid regions for the trapezoidal velocity program with $L/D$ of 0.5, 1.0, 2.0, and 3.0. As in figure 9, most of the entrained fluid initially resides in front of the nozzle exit plane. In fact, as the volume of entrained fluid increases with $L/D$, the fraction of it residing in front of the nozzle increases until for $L/D = 3$ the largest portion of entrained fluid comes from very near the nozzle centerline. This trend may seem at odds with the fact that entrainment into the ring occurs from the rear of the forming spiral and closer to the nozzle lip, but it results from the increased effect of Biot-Savart induction sweeping the entrained fluid over the vortex spiral for larger $L/D$ due to the larger ring circulation and longer formation period as $L/D$ is increased.
Fig. 10 Shape of the entrained fluid regions at $t^* = 0$ for the trapezoidal velocity program with $L/D = 0.5, 1.0, 2.0,$ and $3.0$.

C. Impulse evolution of ejected and entrained fluid

Figure 11 shows the ejected and entrained momentum evolution of a vortex ring from jet initiation to the steadily translating state for the trapezoidal velocity program with $L/D = 2.0$. Axial ejected and entrained momenta ($I_{EJ}$ and $I_E$) were calculated using

\[ I_{EJ} = m_{EJ}U_{EJ} \]
\[ I_E = m_EU_E \]  

where, $U_{EJ}$ and $U_E$ represent the axial velocities of the center of masses of the ejected and entrained fluids, respectively. The results are normalized using $I_T$ computed at $t^* = 2.5$, which is the point where $I_T$ reaches its final, steady value (i.e., the ring is fully developed and boundary effects are minimal). Figure 11 indicates that the bulk of the ejected fluid impulse is obtained during the initial acceleration similar to an isolated rigid particle. Subsequently, $I_{EJ}$ decreases as its momentum is shared with the entrained mass and external fluid (i.e., fluid not in either the ejected or entrained fluid domain). The latter represents what eventually is the added mass component, and is clearly more significant than the entrained mass momentum because $I_{EJ}$
decreases faster than $I_E$ increases for $t^* < 1.0$. The sharp decrease in $I_{EJ}$ during $0.9 < t^* < 1.0$ is because of negative over-pressure at jet termination, which removes momentum from the flow.

After the jet stops (i.e., $t^* > 1.0$), ejected fluid continues to transfer its momentum to the entrained fluid. The exchange between the ejected and entrained fluid components appears as oscillations in the respective components for $t^* = 1.0 - 5.0$ as a result of the spiraling of the two fluid components around each other during the roll-up and entrainment process as illustrated in figure 7.

![Graph showing time variation of ejected and entrained fluid momenta](image)

Fig. 11 Time variation of ejected and entrained fluid momenta for the trapezoidal velocity program with $L/D = 2.0$.

A key observation from figure 11 is that the entrained fluid gains most of its momentum after jet termination. Indeed, at jet termination, the momentum of the entrained fluid only accounts for about 10% of $I_T$. Following jet termination, the fluid motion evolves according to conservation of total momentum, so one may conclude that entrainment contributes very little to any impulse benefit provided by pulsation. On the other hand, this also indicates that the momentum cost to transport entrained fluid with the ring is small.

Figure 12 illustrates the time variation of $I_E$ for the three studied velocity programs. Note that $I_E$ for the triangular PS case is negligible until $t^* = 0.75$. On the other hand, for the
triangular NS case $I_E$ increases throughout the pulse duration. Thus, a rapid initial acceleration (e.g., trapezoidal and triangular NS velocity program) immediately imparts momentum to the entrained fluid, and a gradual acceleration delays this process. Somewhat counter intuitively, Figure 12 also shows that $I_E$ rises rapidly when $U_p(t)$ begins decelerating. This is observation is related to the more rapid fluid entrainment into the ring following jet termination (Olcay and Krueger [26]).

Fig. 12 Time variation of entrained fluid momentum for trapezoidal, triangular PS, and NS velocity programs with $L/D = 2.0$.

The entrained fluid momentum variation for different stroke ratios is given in figure 13. Clearly the maximum $I_E$ is a larger fraction of the total impulse as $L/D$ decreases, except for $L/D = 0.5$. Figure 13 also shows that all of the $L/D$s except for $L/D = 3.0$ makes a dip during jet ejection. This indicates that the entrained fluid for these cases is slowed (or even moves backward as for $L/D = 0.5$, i.e., the entrained fluid gets pushed back during jet start up) by its interaction with the forming ring. During $0.9 \leq t^* \leq 1.0$, a kink appears in figure 13. This is due to the jet termination which pulls the fluid around the nozzle lip. As in figure 12, $I_E$ increases significantly for all $L/D$s following jet deceleration.
Fig. 13 Time variation of entrained fluid momentum for $L/D = 0.5, 1.0, 2.0$ and $3.0$ for the trapezoidal velocity program.

Figure 14 gives summary of $I_E/I_T$ values versus $I_U/(\rho U_M D^3)$ for all the studied cases when the jet is terminated (i.e., $t^* = 1.0$). The value of $I_E$ at this instant is most relevant for assessing the input required from the jet to entrain $m_E$ since the flow evolves according to conservation of momentum for $t^* > 1$. The jet momentum $I_U$ is chosen as the abscissa in figure 14 since it is an indicator of the bulk effect of the jet for a given stroke ratio and velocity program combination. Despite variations in the maximum $I_E$ obtained (see figure 13), figure 14 shows that $I_E/I_T$ at $t^* = 1.0$ is nearly insensitive to the generating conditions for nearly all cases tested except for the triangular NS case. Triangular NS case produces almost two times larger $I_E/I_T$ value than other cases due to the increased rate of entrainment occurring near the beginning of jet initiation for this case. In all cases, however, $I_E/I_T$ is small, indicating the momentum cost of entraining fluid is generally small.
Fig. 14 $I_E / I_T$ versus $I_U/(\rho U_mD^3)$ for the trapezoidal velocity program with $L/D = 0.5, 1.0, 2.0, 3.0$ and triangular NS and PS with $L/D = 2.0$ at $t^* = 1.0$.

Returning to the overall flow evolution, Figure 11 also shows the momentum of the fluid inside the ellipsoidal vortex ring bubble (see figure 7(e)), namely, $I_R = I_{EJ} + I_E$. (The evolution of $I_R$ and $I_{EJ}$ for the other cases simulated can be found in Olcay [25].) This “ring momentum” excludes the momentum associated with added mass. Figure 11 shows that $I_R$ decreases for $t^* \geq 2.5$. This is due to additional ambient fluid entrainment by vorticity diffusion as documented by Maxworthy [20, 21], Dabiri and Gharib [9] and Olcay and Krueger [26]. The fluid entrained for $t^* \geq 2.5$ initially resides in front of the nozzle near the jet centerline as illustrated by the FTLE field in figure 15 (note the ridge near $r/D \approx 0.2$ for $0.0 \leq x/D \leq 4.0$). As the ring translates forward, this fluid is swept over the top of the ring by Biot-Savart induction and entrained into the back of the ring. As this entrainment follows the mechanism described by Maxworthy [20] and Dabiri and Gharib [9] and occurs primarily after the ring is completely formed, we refer to this fluid as fluid entrained by viscous diffusion. Although this additional fluid steals some of the ejected fluid’s momentum, the transfer occurs primarily after the ring is formed since this additional fluid’s velocity is essentially zero until it is encountered
by the ring. Hence, it does not significantly affect our momentum calculations during ring formation (i.e., \(t^* < 2.5\)).

Fig. 15 FTLE fields illustrating entrained fluid in front of the nozzle for the trapezoidal velocity program with \(L/D = 2.0\) at \(t^* = 0.0\).

A key component contributing to the difference between \(I_R\) and \(I_T\) in figure 11 for \(t^* > 1.0\) is added mass. As described by Saffman [32], Krueger [14], Krueger and Gharib [15] and Dabiri [7], added mass is set in motion by the ring but external to the ring boundary. Added mass in this context is well defined only after the ring is formed, at which point it may be computed using standard potential flow analysis for the closed streamline encircling the ring in the frame of reference moving with the ring [18, 22]. In this case, the ring boundary is reasonably well approximated as an ellipsoid. For an ellipsoid with major axis \(a\) and minor axes \(b\) and \(c\), the added mass coefficient is given by

\[
\lambda = \frac{\alpha_0}{2 - \alpha_0}
\]  

(15)

where, \(\alpha_0 = \int_0^\infty \frac{(abc)dz}{(a^2 + z)k_z}\), and \(k_z = \sqrt{(a^2 + z)(b^2 + z)(c^2 + z)}\) [16].

Then, \(\lambda\) is related to the momentum associated with added mass through
\[ I_A = \lambda m_R U_R \]  

where, \( m_R = m_{EJ} + m_E \) is the fluid mass inside the ellipsoidal boundary of the ring (see figure 7(e)). For the case shown in figure 11, \( \lambda = 0.39 \) was computed using equation (15) with \( a \) and \( b = c \) determined as 0.90\( D \) and 0.62\( D \), respectively, from a streamline identifying the vortex ring at \( t' = 2.5 \) (illustrated in figure 16). The calculation was done at \( t' = 2.5 \) because the vortex ring is not formed for \( t' < 2.5 \) and additional ambient fluid is entrained for \( t' > 2.5 \), so \( t' = 2.5 \) gives best accuracy. The computed result for \( \lambda \) assuming an ellipsoidal ring gives \( I_A/I_T = 0.23 \).

![Streamline identifying vortex ring with ejected/entrained fluid domains at \( t' = 2.5 \).](image)

Fig. 16 Streamline identifying vortex ring with ejected/entrained fluid domains at \( t' = 2.5 \).

Based on equation (1) \( I_{EJ} + I_E + I_A = I_T \). From figure 11, \( I_{EJ}/I_T = 0.38 \) and \( I_E/I_T = 0.19 \) at \( t' = 2.5 \), which together with the calculated value for \( I_A/I_T \) only sums to 0.80 rather than 1.0. The discrepancy is in \( I_A/I_T \), which figure 11 shows should be 0.43 at \( t' = 2.5 \), not 0.23 as computed above, indicating that the use of the ellipsoid assumption can have as much as 46 % error even for a formed vortex ring. Recently, Dabiri [7] computed the added mass coefficient of an empirically generated vortex ring (at \( Re_J = 1400 \) and \( L/D = 2.0 \) with an impulsive piston velocity program), and documented \( \lambda \) to be 0.72 for a fully formed vortex ring.
The use of 0.72 for $\lambda$ gives $J_A/J_T = 0.41$ at $t^* = 2.5$ providing reasonable agreement with 0.43 obtained from figure 11.

The magnitude of $I_A$ in figure 11 should not go unnoticed. For the completely formed vortex ring ($t^* = 2.5$) more than 40% of the total impulse is associated with added mass, more than twice the value of the entrained fluid impulse at the same instant. Indeed, by $t^* = 2.5$ almost half of the ejected fluid's momentum is spent to set the added mass into motion. Figure 17 confirms the significance of $I_A$ for the cases studied at $t^* = 2.5$. While all the cases show about half of the total impulse is expended to set the added mass in motion, lowering stroke ratio or gradual deceleration in the velocity program (triangular NS case) can increase the momentum of added mass more than 20%. In the context of the equivalence between equations (1) and (2), added mass clearly makes a much larger contribution to nozzle exit over-pressure ($I_0$) and the associated impulse benefit than entrained mass.

Fig. 17 $I_A/I_T$ versus $I_U/(\rho U_M D^2)$ for the trapezoidal velocity program with $L/D = 0.5, 1.0, 2.0, 3.0$ and triangular NS and PS with $L/D = 2.0$ at $t^* = 2.5$.
To quantitatively assess the influence of the entrained and added mass components on $I_p$ and the associated influence of the ring generating conditions ($L/D$ and velocity program), it is observed that

$$I_p = I_e + I_A + I_{ej} - I_U = (I_e + I_A) - (I_U - I_{ej})$$  \hspace{1cm} (17)

(see equations (1) and (2)). Physically, $I_e + I_A$ is the momentum gained by the ambient fluid and is the resulting "output" of the jet action related to $I_p$. Conversely, $I_U - I_{ej}$ is the jet momentum contribution, or "input, to the ambient fluid momentum gain. Hence, a useful measure of output to input for the ambient fluid momentum contribution to $I_p$ is

$$\eta = \frac{I_e + I_A}{I_U - I_{ej}}.$$ \hspace{1cm} (18)

Based on equation (17), it is desirable to maximize the performance metric $\eta$. In particular, maximizing $\eta$ maximizes the pressure impulse per unit of jet momentum contributed to the acceleration of ambient fluid ($I_p/(I_U - I_{ej})$). Figure 18 plots $\eta$ evaluated at $t^* = 2.5$ for all cases considered. From figure 18 it is observed that $\eta$ increases as $L/D$ is increased, indicating that higher a stroke ratio provides a stronger relative contribution to pressure impulse and ambient fluid momentum. The rate of increases slows, however, as $L/D$ increases and the formation number [11] is approached, consistent with the results of Krueger and Gharib [15]. Comparing the open symbols shows that the trapezoidal and triangular PS velocity programs exhibit higher $\eta$ values compared to the triangular NS velocity program for the same $L/D$, indicating that rapid deceleration velocity programs provide higher ambient fluid impulse per amount of momentum transferred from ejected fluid. This result is somewhat surprising as a rapid initial acceleration might seem to have the stronger effect as it promotes rapid acceleration of the ambient fluid. The rapid deceleration of the trapezoidal and triangular PS velocity
programs, however, prevents as much $I_U$ from being used to accelerate this fluid and $\eta$ is higher for these cases. In any case, the effect of velocity program on $\eta$ is small for the cases considered.

![Graph](image)

Fig. 18 $\eta$ at $t^* = 2.5$ versus $I_U/(\rho U_M D^3)$ for all cases considered.

**IV. Conclusions**

Ejected and entrained impulse calculations were performed to investigate each term’s contribution to the formed ring impulse. CFD combined with LCS were utilized to identify initial fluid boundaries. Then, these fluid boundaries were advected via 4th order Runge-Kutta integration scheme. The effect of velocity programs and $L/D$s on the evolution and momentum of the ejected and entrained fluid domains were investigated.

The velocity program used to generate a single jet pulse played a key role in the $I_E$ and $I_A$ results. Particularly, $I_E$ was low while the jet was on indicating that most of the ejected fluid’s momentum was transferred to the added mass during this phase and that the momentum input required from a starting jet to entrain ambient fluid is small. Most of the momentum gained by the entrained fluid occurred after the jet began to decelerate. This constrained $I_E$ at
jet termination \((t^* = 1.0)\) to nearly the same value for all but the triangular NS case, which initiated deceleration very early in the jet pulse. Once the jet was terminated, both \(I_E\) and \(I_{EJ}\) oscillated as the ejected and entrained fluid rotated about the center of vorticity in the ring. On the other hand, it was observed that \(I_A\) captured about 50% of total impulse of all the cases studied when the ring was formed. It was also determined that lower \(L/Ds\) and/or gradual deceleration in the velocity program could increase the momentum of added mass more than 20% indicating that the effect of velocity program and associated stroke ratio can be significant in the determination of added mass momentum. Generally changes in the velocity program had a much larger effect than might otherwise be anticipated since the jet impulse and circulation based on the bulk jet velocity (i.e., from the slug model) were the same for the triangular NS and PS programs.

Since \(I_p = I_E + I_A - (I_U - I_{EJ})\), \(\eta = (I_E + I_A)/(I_U - I_{EJ})\) was used as a performance metric to gage the effectiveness of different cases in contributing to \(I_p\) as a function of the vortex formation process. It was observed that the rate of increase of \(\eta\) slows as \(L/D\) increases and the formation number [11] is approached. It was also observed that the triangular PS velocity program and higher \(L/D\) provided a larger \(\eta\), indicating a larger contribution to \(I_p\) for a given input \(I_U - I_{EJ}\) in these cases.

An additional benefit of this study is the identification of the initial shape and location of entrained fluid around the nozzle exit plane. This information can be useful for promoting efficient mixing of two reactive substances by indicating where a regent should be concentrated or what formation parameters (velocity programs or \(L/D\)) should be adjusted to enhance the
process. By adjusting formation parameters (e.g., velocity program and $L/D$) mixing rate can be determined based on the application.

While this study only investigates the impulse variation of starting jets, the many of the general trends in $I_\varepsilon$ and $I_\lambda$ can be applicable to pulsed jets where the time duration between the pulses is long enough to avoid strong vortex interaction between pulses [17]. Often applications involving pulsed jets such as pulsed jet propulsion (Nichols et al. [23], Bartol et al. [3, 4]), the jet pulses are widely separated so that the flow between pulses is nearly undisturbed. The present results can be expected to be representative of the trends observed in such cases.

REFERENCES


2. Ball, R.S.: Account of experiments upon the resistance of air to the motion of vortex rings. Philosophical Magazine 342, 208-210 (1871).


