SABBATICAL REPORT
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cc: The Improvement of Learning Committee
The report outlines activities and accomplishments of the author's sabbatical leave during Fall Semester of 2008.

The main objectives of my sabbatical leave were the following: > to identify and collect appropriate and up-to-date material for a book on Vehicle System Dynamics which I have been solicited to write, > to visit finest European academic research centers, > and to prepare a general outline and detailed contents of the book. In order to accomplish the objectives of my sabbatical leave I have been engaged in the following activities:

• A good deal of my sabbatical leave I spend in Europe visiting three European leading academic centers and working on a book on Vehicle System Dynamics. In early June of 2008, I visited Technical University of Berlin (TUB) which is one of Germany's premier teaching and research centers. In 1980 I was a Research Associate at TUB and this afforded me excellent access to its facilities. Further, I spent some time with experts on vehicle system dynamics at TUB. Scholarly discourse with those people provided tremendous impetus to my publication effort on vehicle system dynamics.

• In mid-June of 2008, I visited one of the finest European technical universities, namely, Delft University of Technology (DUT) in The Netherlands. During my visit at DUT I spent most of the time in the Vehicle Research Laboratory at the Department of Mechanical Engineering. It is a well known academic research center in Europe having state-of-the-art research facilities. In 1980-1981 I was a Research Associate at DUT and this afforded me excellent access to its faculty and facilities.
The time which I spent at the Vehicle Research Laboratory enabled me to exchange ideas with experts on vehicle system dynamics and was very useful to my publication efforts on Vehicle Dynamics. Also, during my visit in the Department of Mechanical Engineering at DUT, I intensely explored teaching methodology in the areas of design and system dynamics which are my primary responsibilities in the Department of Mechanical & Industrial Engineering at UW-Platteville.

- In July of 2008 I visited another fine European Technical University, namely, Technical University of Wroclaw, Poland where I spent two weeks. This is one of Poland's premier teaching and research centers. The Department of Mechanical Engineering at the Technical University of Wroclaw offers undergraduate and graduate degrees at the main campus and undergraduate degree at three branches. At present there are 1200 students with over 90 Faculty. The curriculum offered by the Department of Mechanical Engineering is very rich and diversified. It is particularly strong in the area of mechanical system design, manufacturing and control. The teaching and research laboratories are well equipped and modern. Since I spent two weeks at the Technical University of Wroclaw (TUW) I had enough time to closely explore the Mechanical Engineering curriculum and to learn about their teaching methodology. Also, while at TUW, I gave a seminar on Basic Linear Theory of Handling and Stability of Automobiles.

The faculty at the academic institutions which I visited were most receptive to my visits and the exchange of professional dialog. At all three places, I intensely explored teaching methodologies in the area of mechanical design, and research activities in the area of vehicle system dynamics. Also during my visit at Technical University of Berlin, Delft University of Technology and Technical University of Wroclaw, I participated in several meetings with key faculty in sharing ideas on engineering design education.

The benefits which accrued from my sabbatical can be summarized as follows:

- My interactions with the faculty at Technical University of Berlin, Delft University of Technology and at the Technical University of Wroclaw acquainted them with UW-Platteville and the nature of our Mechanical Engineering program.

- During my visits at Technical University of Berlin, Delft University of Technology and Technical University of Wroclaw I was engaged in intensive study of the teaching methodology in the design area at these institutions.
• The knowledge I gained of European pedagogical approaches to student design projects will allow me to increase learning outcomes and the quality of design education in my own courses at UW-Platteville.

• Further, my sabbatical facilitated the development of the organizational structure and content of my anticipated book on Vehicle System Dynamics as outlined in this document.

• The time spent away has rejuvenated me in mind, body, and spirit for the teaching profession which I embraced as my life's career choice.

• To share the results of my professional activities during sabbatical leave with university colleagues, that is, with Dr. Michael Momot, Dr. David Kraemer, and Dr. David Kunz, I invited them to write together a paper on vehicle system dynamics. The paper entitled BASIC LINEAR THEORY OF HANDLING AND STABILITY OF AUTOMOBILES, Authors: Stan Lukowski, Ph.D.; Michael Momot, Ph.D.; David Kraemer, Ph.D.; David Kunz, Ph.D., was submitted and has been accepted for publication (please see the attached letter from the publisher and a copy of the paper) in Proceedings of the Institution of Mechanical Engineers (I MECH E), Part D, Journal of Automobile Engineering. The I MECH E is the most prestigious and well known engineering science publishing organization based in London, UK.

• Also, together with my colleagues Dr. Dave Kuntz and Dr. Jeff Hoerning from the Department of Mechanical & Industrial engineering, I have written another paper entitled GEOMETRY OF CONTACT AND HERTZIAN STRESS ANALYSIS OF FRICTIONAL COUPLING ELEMENTS OF MULTIDISK STEPLESS TRANSMISSION WITH INITIAL LINE CONTACT. The paper has been submitted for the 2009 Power Transmission and Gearing Conference, which will be held in conjunction with the ASME International Design Engineering Technical Conferences at the San Diego Convention Center, in San Diego, California, from August 30-September 2, 2009.

GENERAL OUTLINE AND DETAILED CONTENTS OF THE BOOK ENTITLED:
FUNDAMENTALS OF VEHICLE SYSTEM DYNAMICS

This book is intended to be used as a text and/or as a reference for a course on vehicle system dynamics. It will present the foundation of the analytical studies of motor vehicle handling and performance. The development of the governing equations of vehicle motion and computer implementation in the solution of these equations will be emphasized. Analysis of vehicle subsystems such as tire/road interface, wheel rotational dynamics, and suspension kinematics
and compliance will be covered along with the analytical methods for predicting many aspects of road vehicle performance, handling, and safety. The foundation for the book which I wish to author would be class notes I have assembled while teaching courses on vehicle system dynamics at Wayne State University in Detroit, Technical University of Wroclaw, Poland, and The University of Akron in Ohio as well as the material which I have collected during my sabbatical leave. The following is a tentative outline of the contents of this book:

1. INTRODUCTION - An Overview of the Subject

2. DYNAMIC EQUATION FORMULATION
   2.1 Basic Steps in Modeling and Analysis of Dynamic Systems
   2.2 Background Material for System Modeling and Analysis
   2.3 Newtonian and Lagrangian Mechanics

3. VEHICLE DYNAMICS TERMINOLOGY
   3.1 Basic Properties of the Tire-Vehicle System: Overview and Definitions
   3.2 Performance Regimes
   3.3 Terminology and Symbols

4. TIRE MECHANICS
   4.1 Tire Traction Models
   4.2 Friction Circle Concept
   4.3 Tire Traction Model Incorporating Interactions Between Lateral and Longitudinal Slip

5. SUSPENSION KINEMATICS AND COMPLIANCE
   5.1 Typical Suspension Design
   5.2 Kinematic Properties of the Double-Wishbone Suspension
   5.3 Wheel Inclination Produced by Jounce-Rebound Motions
   5.4 Location of the Roll-Center
   5.5 Wheel Inclination Resulting from Body Roll
   5.6 Suspension Spring and Damper Forces

6. SIMPLE HANDLING VEHICLE MODELS
   6.1 Dynamic Behavior of a Vehicle Under Steady-State Conditions
   6.2 Vehicle Model and Equations of Motion
   6.3 Computer Implementation and Numerical Results
6.4 Vehicle Directional Response to Front Wheel Steering Input at Varying Forward Velocity
6.5 Handling Diagrams
6.5 Under-steer, Over-steer, and Neutral-steer Vehicle
6.6 Vehicle Directional Stability

7. VEHICLE MODEL INCORPORATING SUSPENSION DYNAMICS AND COMPLIANCE
   7.1 Formulation of Governing Equations
   7.2 Special Lagrange Equations
   7.3 Modified Equations of Vehicle Motion
   7.4 Tire Normal Load Transfer Due to Body Roll
   7.5 Tire Normal Load Transfer Due to Pitch Motion
   7.6 Steering System Dynamics
   7.7 Wheel Rotational Dynamics
   7.8 Longitudinal and Lateral Tire Slip

8. COMPUTER IMPLEMENTATION AND NUMERICAL RESULTS
   8.1 Vehicle Transient and Steady-State Turning Response as Influenced by Front-Rear Load Distribution
   8.2 Vehicle Transient and Steady-State Turning Response as Influenced by Tire Traction Parameters
   8.3 Vehicle Transient and Steady-State Turning Response as Influenced by Drive Arrangement (Front vs. Rear)

9. BRAKING PERFORMANCE
   9.1 Basic Equations
   9.2 Braking Forces
   9.3 Braking Efficiency
   9.4 Rear Wheel Lockup
   9.5 Front Wheel Lockup
   9.6 ABS Performance
Basic linear theory of handling and stability of automobiles

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Abstract: The directional response and stability characteristics of the motor vehicle are examined by considering the automobile to be a mechanical system which is described by linear differential equations with constant coefficients. The directional response behaviour and stability characteristics of the motor vehicle are important performance modes of vehicle operation often equated with handling. The objective of this analysis is to show how steady state turning behaviour depends upon various vehicle design factors and motion variables. A family of characteristic handling diagrams are obtained for linear and non-linear ranges of tyre operation. The handling diagrams show the dependence of the vehicle’s directional behaviour upon the tyre lateral force characteristics, the front and rear load distributions, and the vehicle’s forward speed. Information obtained from this study of vehicle response to steering control gives insight into the turning behaviour of a real vehicle and is useful for safety objectives.

Keywords: basic linear theory, handling, stability characteristics, directional response

1 INTRODUCTION

The purpose of this study is to develop analytical methods that are useful for examining or predicting the dynamic behaviour of a motor vehicle subjected to control inputs. The term ‘dynamic behaviour’ should be understood as vehicle performance, i.e., the response behaviour in the lateral direction exhibited by the motor vehicle subjected to any control inputs or external disturbances, either singly or in combination.

The directional response behaviour and stability characteristics of the motor vehicle are important performance modes of vehicle operation often equated with handling. Handling is a loosely used term meant to imply the responsiveness of a vehicle to driver input, or the ease of control. As such, handling is an overall measure of the vehicle–driver combination. The driver and vehicle form a closed-loop system, meaning that the operation of an automobile involves interaction between vehicle, driver, and the road. For purposes of characterizing only the vehicle, open-loop behaviour is considered here in this study. The term open loop refers to vehicle response to specific steering inputs and is more precisely defined as directional response behaviour [1, 2].

2 VEHICLE MODEL AND EQATIONS OF MOTION

2.1 Vehicle model description

In this study, attention is focused on more fundamental aspects of vehicle dynamic behaviour. For this purpose, a simplified one-mass vehicle model that is free to yaw and sideslip while negotiating a turn with a constant speed is examined. The vehicle model as shown in Fig. 1 does not have body roll and load transfer. The driving torque applied to the drive wheels that is required to keep the vehicle speed constant is assumed to be small, permitting all wheels to be treated as ‘free rolling’ such that the tyre lateral force depends only on the tyre lateral slip. As a result of small-disturbance assumptions (made for the purposes of linearization) it can be concluded...
that the variation in longitudinal forces during directional motion is negligible. Consequently, the vehicle does not experience any longitudinal accelerations, provided that the driving torque is in equilibrium with resistance to forward motion. The resistance develops primarily from the aerodynamic drag and rolling resistance forces produced by the tyres. The control inputs to the vehicle model consist of relatively small steering-wheel angular displacements. Thus, small steering inputs do not cause any change in the vehicle’s forward velocity with the result that linearized treatment of vehicle motion analysis implies constant velocity manoeuvres.

The above limitations concerning the vehicle model are necessary for the analysis to remain relatively simple and easy to comprehend. These limitations will also allow us to determine the primary factors influencing vehicle directional behaviour.

2.2 Formulation of governing equations

In order to derive the governing equations of motion of the vehicle model, either the Newton–Euler approach or the methods of analytical mechanics can be employed. In this report, it proves to be particularly convenient to use the latter procedure, namely the Lagrange equations. However, a minor complication results from the fact that customary Lagrange equations written in terms of generalized coordinates only yield meaningful results when the generalized coordinates are also inertial or true coordinates. Mathematically, a coordinate system may be considered ‘true’ if integration of the body’s velocity vectors with respect to time yields the corresponding location coordinates.

Unfortunately, expression of the vehicle motion in terms of a fixed inertial coordinate system when the vehicle is undergoing simultaneous translation, yaw, and/or roll motions is very cumbersome. To circumvent this difficulty, the vehicle motions can be expressed in terms of a moving coordinate system which translates and yaws with the vehicle. However, these moving coordinates will not be inertial and thus will not satisfy the integrability requirement of true coordinates. Therefore, application of the customary Lagrange equations to a moving coordinate system will result in erroneous equations of motion.

A method does exist, however, by which the Lagrange equations may be correctly derived for a moving coordinate system. This method requires the use of so-called quasi-coordinates, and the Lagrange equations expressed in terms of these coordinates are called the special lagrange equations [3, 4].

2.3 Special Lagrange equations

For the model of the vehicle being considered, the special lagrange equations have the form
\[
\frac{dT}{du} - r \frac{dT}{dv} = \sum F_x \tag{1}
\]
\[
\frac{d}{dt} \left( \frac{dT}{du} \right) + r \frac{d}{dt} \left( \frac{dT}{dv} \right) = \sum F_y \tag{2}
\]
\[
\frac{d}{dt} \left( \frac{dT}{dr} \right) - \frac{dT}{du} + u \frac{dT}{dv} = \sum M_z \tag{3}
\]

A detailed derivation of these equations can be found in reference [4].

The total kinetic energy of the vehicle model may be written in terms for the translational and rotational (yawing) velocities, such that

\[
T = \frac{1}{2} m V^2 + \frac{1}{2} I_{zz} r^2 \tag{4}
\]

The translational velocity \( V \) consists of \( u \) and \( v \) components and may be written as

\[
V^2 = u^2 + v^2 \tag{5}
\]

Substituting for the translational velocity from equation (5) into equation (4), the total kinetic energy of the considered vehicle model may then be written as

\[
T = \frac{1}{2} m (u^2 + v^2) + \frac{1}{2} I_{zz} r^2 \tag{6}
\]

Evaluating the terms in equations (1) to (3) yields

\[
\frac{dT}{du} = m \dot{u} \tag{7}
\]

\[
\frac{d}{dt} \left( \frac{dT}{du} \right) = m \ddot{u} \tag{8}
\]

\[
\frac{dT}{dv} = m \dot{v} \tag{9}
\]

\[
\frac{d}{dt} \left( \frac{dT}{dv} \right) = m \ddot{v} \tag{10}
\]

\[
\frac{dT}{dr} = I_{zz} \dot{r} \tag{11}
\]

\[
\frac{d}{dt} \left( \frac{dT}{dr} \right) = I_{zz} \ddot{r} \tag{12}
\]

Substituting the appropriate partial derivatives from equations (7) to (12) into equations (1) to (3) yields three differential equations of motion

\[
m(\dot{u} - rv) = \sum F_x \tag{13}
\]

\[
m(\dot{v} + ru) = \sum F_y \tag{14}
\]

\[
I_{zz} \ddot{r} = \sum M_z \tag{15}
\]

Equations (13) to (15) form a set of three differential equations which describe the motion of a vehicle with longitudinal, lateral, and yaw degrees of freedom.

2.4 Modified equations of vehicle motion

Since the yaw and sideslip velocities are known to be small compared with the steady state forward velocity of the vehicle, the products of these terms (i.e. \( rv \)) may be considered negligible. On the basis of these assumptions, the three coupled differential equations of motion of the vehicle model employed in this analysis are obtained as

\[
\sum F_x = m \dot{u} \tag{16}
\]

\[
\sum F_y = m(\dot{v} + ru) \tag{17}
\]

\[
\sum M_z = I_{zz} \ddot{r} \tag{18}
\]

Note that a longitudinal acceleration term \( \ddot{u} \) appears only in the first equation of the above three equations of motion. If it is assumed that the forward speed \( u \) is kept constant, then \( \ddot{u} = 0 \). Consequently, the first equation can be ignored and it can be concluded that the remaining two equations apply to a one-mass motor vehicle which is moving with a constant velocity \( u \) along its longitudinal axis. Thus, the linearized equations of motion for the assumed vehicle model are

\[
\sum F_y = m(\dot{v} + ru) \tag{19}
\]

\[
\sum M_z = I_{zz} \ddot{r} \tag{20}
\]

3 VEHICLE HANDLING AND STABILITY UNDER STEADY STATE CONDITIONS

3.1 Derivation of the handling diagram for the linear range of tyre operation

For steady state conditions, it may further be assumed that the lateral and yawing accelerations...
are zero, i.e. \( \dot{\alpha} = \dot{r} = 0 \). Also, for a small front-wheel steer displacement, it can be assumed that \( u = V \). Under these assumptions, equations (19) and (20) become

\[
\sum F_y = mVr \tag{21}
\]

\[
\sum M_z = 0 \tag{22}
\]

where

\[
\sum F_y = F_{y1} + F_{y2} + F_{y3} + F_{y4} \tag{23}
\]

\[
\sum M_z = a(F_{y1} + F_{y2}) - b(F_{y3} + F_{y4}) \tag{24}
\]

The lateral force components of the tyres along the vehicle axes are computed from

\[
\sum F_{yi} = F_{yi} \cos(\delta_i) \tag{25}
\]

where the steering angles \( \delta_i \) (\( i = 1, 2, 3, 4 \)) are

\[
\delta_1 = \delta_2 = \delta \tag{26}
\]

\[
\delta_3 = \delta_4 = \delta \tag{27}
\]

For small front-wheel steer displacement \( \delta \), it can be assumed that \( \cos(\delta) \approx 1 \). Thus equation (25) becomes

\[
\sum F_{yi} = F_{yi} \tag{28}
\]

Note that each tyre on the vehicle is side slipping and turning on a curved path of radius \( R \). Ignoring the lateral distortion of the tyre due to path curvature and assuming that the slip angles remain small such that the cornering force is linearly related to the slip angle (Fig. 2), for a linear range of tyre operations, the lateral tyre forces generated at the tyre–road interface oriented with respect to the wheel plane can be computed from

\[
F_{yi} = \left( \frac{\partial F_y}{\partial \alpha} \right) \alpha_i \tag{29}
\]

where \( \frac{F_{yi}}{\alpha_i} \) is the cornering stiffness of the \( i \)th tire (\( i = 1, 2, 3, 4 \)) and \( \alpha_i \) is the slip angle of the \( i \)th tire (\( i = 1, 2, 3, 4 \)).

In a steady turn \( (u, v, \text{ and } r \text{ are fixed quantities}) \) the lateral velocities \( v_1 \) and \( v_2 \) at the front and rear axles respectively, are given by (see Fig. 1)

\[
v_1 = v + ar \tag{30}
\]

\[
v_2 = v - br \tag{31}
\]

In this instance, it is found that the angles between the velocity vector of each tyre and the tyre centre plane are as in Fig 1. Note that, if \( u \gg v + ar, \) \( u \gg v - br, \) and \( \beta \) is a small angle, the front and rear slip angles may be expressed as

\[
\alpha_1 = \alpha_{1,2} = \frac{v}{u} + \frac{ar}{u} - \delta = \beta + \frac{ar}{u} - \delta \tag{32}
\]

\[
\alpha_2 = \alpha_{3,4} = \frac{v}{u} - \frac{br}{u} = \beta - \frac{br}{u} \tag{33}
\]

Denoting \( (F_{yi}/\alpha_i) |_f \) and \( (F_{yi}/\alpha_i) |_r \) as the resultant lateral stiffness of both front and both rear tyres respectively, the equations can be written as

\[
F_{y1,2} = -\frac{\partial F_y}{\partial \alpha} \bigg|_f \alpha_1 \tag{34}
\]

\[
F_{y3,4} = -\frac{\partial F_y}{\partial \alpha} \bigg|_r \alpha_1 \tag{35}
\]

With the aid of equations (30) to (35), the equations of equilibrium are obtained as

\[
\left( \frac{\partial F_y}{\partial \alpha} \bigg|_f + \frac{\partial F_y}{\partial \alpha} \bigg|_r \right) \beta + \left( \frac{a \partial F_y}{u} \bigg|_f - \frac{b \partial F_y}{u} \bigg|_r + mV \right) r = \frac{\partial F_y}{\partial \alpha} \bigg|_f \tag{36}
\]
(a \frac{\partial F_y}{\partial \alpha} |_t - b \frac{\partial F_y}{\partial \alpha} |_t) \beta + (a^2 \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t) r = a \frac{\partial F_y}{\partial \alpha} |_t \delta \tag{37}

Equations (36) and (37) can be expressed in matrix form as

\[
\begin{bmatrix}
\frac{\partial F_y}{\partial \alpha} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t & a \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + mV \\
\frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t & \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t
\end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} = \begin{bmatrix} \frac{\partial F_y}{\partial \alpha} |_t \delta \\ \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t \delta \end{bmatrix}
\tag{38}
\]

Solving the above equation for the yawing velocity \( r \) with the aid of the Cramer rule gives

\[
r = \frac{\frac{\partial F_y}{\partial \alpha} |_t - \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t}{\frac{\partial F_y}{\partial \alpha} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t}
\tag{39}
\]

On evaluating equation (39) for the yawing velocity \( r \), the expression for the ratio of the steady yaw rate to the front-wheel steer displacement \( \delta \) is obtained as

\[
r = \frac{\frac{\partial F_y}{\partial \alpha} |_t - \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t}{\frac{\partial F_y}{\partial \alpha} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t} \tag{40}
\]

where

\[D = \left( a \frac{\partial F_y}{\partial \alpha} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t \right) \left( a \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t + mV \right)
\]

The ratio \( r/\delta \) is called the yaw rate gain. Since \( V = rR \), the yaw rate gain can be transformed to a path curvature gain by noting that

\[
\frac{1}{R} \delta = \frac{1}{V} \frac{r}{\delta}
\tag{41}
\]

Also noting that \( u \approx V \) and with the aid of equation (41), equation (40) is transformed to the relation

\[
\frac{1}{R} \delta = \frac{1}{(a+b)\frac{\partial F_y}{\partial \alpha} |_t - (a \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t)} - mV^2 \tag{42}
\]

Dividing the right-hand side terms of equation (42) by the numerator and substituting for \( a + b = l \), the expression for the path curvature gain reduces to

\[
\frac{l}{R} \delta = \frac{1}{1 - mV^2 \frac{\text{YM}}{\text{FC}_y}}
\tag{43}
\]

where the term

\[\text{YM} = a \frac{\partial F_y}{\partial \alpha} |_t - b \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t
\tag{44}\]

is called, in this analysis, the yawing moment coefficient and

\[\text{FC}_y = \frac{\partial F_y}{u \frac{\partial \alpha}{\partial t}} |_t
\tag{45}\]

is the product of the cornering stiffnesses of the front and rear tyres. Inversely, equation (43) can be written as

\[
\frac{\delta}{l/R} = 1 - mV^2 \frac{\text{YM}}{\text{FC}_y}
\tag{46}\]

or the steer angle required to travel in a steady turn of radius \( R \) is

\[
\delta = \frac{l}{R} \left( 1 - mV^2 \frac{\text{YM}}{\text{FC}_y} \right)
\tag{47}\]

Equation (46) gives the variation in steer angle \( \delta \) that is required for a given vehicle to negotiate a turn of fixed radius \( R \). Figure 3 presents the ratio \( \delta/(l/R) \) plotted against the vehicle velocity \( V \).

Figure 3 shows the variation in steer angle that is required for the vehicle to move in a constant-radius curve when the speed is increased. Three different kinds of vehicle are considered: oversteer (YM > 0), neutral steer (YM = 0), and understeer (YM < 0).

For an understeer vehicle, the required steer angle increases with increasing speed. The figure indicates that for an oversteer vehicle the required steer angle diminishes with increasing speed. In the latter case, the required steer angle changes sign at a speed termed the critical speed. The expression for the critical speed can be obtained by noting that the ratio \( \delta/(l/R) \) at this speed equals zero. Thus, equating the left-hand term of equation (46) to zero and solving for \( V_{\text{cr}} \), gives

\[
V_{\text{cr}} = \sqrt{\frac{\text{CY}}{m \text{YM}}}
\tag{48}\]
where $YM$ and $C_y$ are defined by equations (44) and (45) respectively.

Again consider equation (46) which can also be expressed as

$$\delta = \frac{1}{R} - \frac{l}{R} \frac{MV^2}{FC_y} \tag{49}$$

or as

$$\frac{l}{R} - \delta = \frac{WYMV^2}{C_y gR} \tag{50}$$

On the basis of equations (32) and (33), it is found that the difference between the front and rear slip angles is given by

$$z_f - z_r = \frac{R}{u} - \delta \tag{51}$$

For small sideslip angle, the path curvature in a steady turn is given by $1/R = r/V = r/u$. The substitution of $r/u = 1/R$ into equation (51) yields

$$z_f - z_r = \frac{R}{R} - \delta \tag{52}$$

Thus, the vehicle force and moment balance for a given steer angle (i.e. equation (50)) can be related to the difference between the front and rear slip angles as given by equation (52) and Fig. 3 can be transformed to Fig. 4. In Fig. 4, the new ordinate represents the difference $z_f - z_r$ between the front and rear slip angles. The new abscissa represents the centripetal acceleration $V^2/(gR)$ in units of acceleration $g$ due to gravity.

The handling diagram that it is desired to derive is obtained by rotating the plot of Fig. 4 through $90^\circ$ and combining it with a diagram for centripetal acceleration against path curvature $l/R$. The space of the latter diagram can be filled with lines of constant velocity and lines of constant radius.

Figure 5 represents the combined diagrams for three vehicles with the same mass and wheelbase, but with different handling characteristics. The figure shows that for a given handling characteristic (thick line) the steer angle $\delta$ required for negotiating a certain manoeuvre characterized by $R$ and $V$ can be read directly from the diagram. In the linear range of the handling regime, the steer angle $\delta$ changes linearly with lateral acceleration at a given radius. When the yawing coefficient $YM$ is negative, i.e. $YM < 0$, the steer angle increases with increasing lateral acceleration (i.e. with increasing speed for a constant-radius test), and the vehicle is described as understeering. For $YM > 0$, the vehicle is described as oversteering. For $YM = 0$, the vehicle is said to exhibit neutral steer.

### 3.2 Derivation of the handling diagram for the non-linear range of tyre operation

The steady state turning behaviour of the vehicle discussed so far is limited to the case when tyres operate at a small slip angle. As shown in Fig. 2, at

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**Fig. 3** Required steer angle versus vehicle velocity squared

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small slip angles, the relationship between the tyre lateral forces $F_y$ and slip angle $\alpha$ is linear. At larger slip angles, the tyre operates in the non-linear range. When the vehicle model is extended to include non-linear characteristics, equation (49) can no longer be used to relate the steer angle $\delta$ to the force and moment balance on the vehicle. Now a graphical method for obtaining a handling diagram for the non-linear range of tyre operation will be developed. The graphical method presented in this section is based on work performed by Pacejka [5].

Assuming a fixed set of tyre characteristics which do not change during the motion, equations (21) and (22) can be expressed as

$$\sum F_y = mV^2 \frac{1}{R}$$

$$aF_{yd} = bF_{yr}$$

Fig. 4 Difference between the slip angles versus lateral acceleration

Fig. 5 Handling diagram for the linear range of tyre operation
where

\[ \sum F_y = F_{yf} + F_{yr} \]
\[ F_{yf} = F_{y1} + F_{y2} \]
\[ F_{yr} = F_{y3} + F_{y4} \]
\[ \frac{1}{R} = \frac{r}{V} \]

Solving equations (53) and (54) for \( F_{yf} \) and \( F_{yr} \) gives

\[ F_{yf} = mV^2 \frac{1}{R} \frac{b}{l} \]
\[ F_{yr} = mV^2 \frac{1}{R} \frac{a}{l} \]

From equations (55) and (56),

\[ mV^2 \frac{1}{R} \frac{b}{l} = \frac{F_{yf}}{F_{zd}} = \frac{F_{yr}}{F_{zr}} \]

(57)

Normalizing the relationships given by equation (57) with respect to the vehicle weight \( mg \) yields

\[ \frac{V^2}{gR} = \frac{F_{yf}}{mg(b/l)} = \frac{F_{yr}}{mg(a/l)} \]

(58)

Under the assumption that there is no load transfer, it may be observed that the terms \( mg(b/l) \) and \( mg(a/l) \) represent the resultant vertical loads \( F_{zd} \) and \( F_{zr} \) on the front and rear tyres respectively. The term \( V^2/(gR) \) is the lateral acceleration. Therefore,

\[ \frac{V^2}{gR} = \frac{F_{yf}}{F_{zd}} = \frac{F_{yr}}{F_{zr}} = \frac{F_{yf}}{F_{yf}} = \frac{F_{yr}}{F_{yr}} \]

(59)

Figures 6(a) and 7(a) show the lateral force characteristics of pneumatic tyres (front and rear) normalized with respect to normal loads which occur on a dry road surface, which are denoted in these figures as \( FYNF = FYNR = V^2/(gR) \), where \( FYNF = F_{yf} = F_{zf} / F_{zd} \) and \( FYNR = F_{yr} = F_{zr} / F_{zd} \). According to equation (55), these two tyre curves may be merged into one diagram with the same ordinate \( V^2/(gR) \). For a certain vehicle velocity \( V \), a front slip angle \( \alpha_f \) and a rear slip angle \( \alpha_r \) may be read from this diagram. Subtracting the normalized tyre characteristics from each other in the horizontal direction, a diagram may be constructed with ordinate \( V^2/(gR) \) versus \( \alpha_r - \alpha_f \). This produces the handling curves shown in Figs 6(b) and 7(b) which relate the difference between the slip angles and the lateral acceleration. It can be observed that the required steer angle at a constant \( R \) and \( V \) varies in a non-linear manner with increasing lateral acceleration \( V^2/(gR) \). According to the definition of oversteer and understeer used for the linear vehicle model, Fig. 6(b) \( (\alpha_r > \alpha_f) \) represents an oversteer vehicle, and Fig. 7(b) \( (\alpha_r < \alpha_f) \) an understeer vehicle. Since the slopes of these handling curves do not change in sign as the lateral acceleration increases, the steer angle required to maintain a certain path curvature keeps either decreasing or increasing with increasing speed. Since it is really the variation in steer angle which determines the handling quality of a vehicle,
the cases of Figs 6(b) and 7(b) may indeed be considered as oversteer and understeer respectively.

Alternative definitions of ‘understeer’ and ‘oversteer’ refer to the sign of the slope of the handling curve. Referring to the handling diagrams shown in Figs 6(b) and 7(b), the equivalent definitions for oversteer and understeer vehicle are

\[
\text{Oversteer : } \frac{\partial [V^2/(gR)]}{\partial (\alpha_f - \alpha_r)} < 0
\]

\[
\text{Understeer : } \frac{\partial [V^2/(gR)]}{\partial (\alpha_f - \alpha_r)} > 0
\]

It can be concluded that the neutral-steer vehicle can be defined as

\[
\frac{\partial [V^2/(gR)]}{\partial (\alpha_f - \alpha_r)} = 0
\]

4 SUMMARY AND CONCLUSIONS

An analytical study has been performed to investigate the directional response behaviour of a motor vehicle in steady state turning manoeuvres. This important performance mode of vehicle operation is termed vehicle handling.

Based on this study, a family of characteristic handling diagrams for different tyre lateral force characteristics was obtained. The primary handling regime, i.e. for linear range of tyre operation, is the first stage and is adequately represented by linear relationships. The handling diagrams were obtained from the linearized vehicle model and show the dependence of the vehicle’s directional behaviour upon the lateral tyre force characteristics, the front and rear load distributions, and the vehicle’s forward speed. The primary factor which defines the vehicle-handling character is the value of the coefficient YM which determines the location of the so-called neutral steer point. The forward centre-of-gravity locations, i.e. \( a < b \), and the fact that the rear tyres’ cornering stiffnesses are higher than the front tyres’ cornering stiffnesses, i.e. when \( (F_y/a)_r > (F_y/a)_f \), result in a negative value of YM. In this instance, the vehicle is defined as an understeer. Rearward centre-of-gravity locations, i.e. \( a > b \), and the fact that the front tyres’ cornering stiffnesses are higher than the rear tyres’ cornering stiffnesses, i.e., \( (F_y/a)_f > (F_y/a)_r \), result in a positive value of YM. In this case, the vehicle is said to be an oversteer. It is clear that vehicles with the centre of gravity near the centre of the wheelbase and with the front and rear cornering stiffnesses approximately equal have the value YM = 0 and are characterized as neutral-steer vehicles.

The secondary handling regime which deals with the non-linearities of tyre lateral force characteristics becomes more complex. In general, it cannot be represented by simple equations. A great variety of handling curves can result owing to differences between normalized characteristics of front and rear tyres that derive from differences in the elastic and frictional properties of tyres. It may be concluded that differences in the tyre slip angle give sufficient information about the steering character of an automobile when high lateral accelerations are involved.
REFERENCES


APPENDIX

Notation

\( a \) distance between the front axle and the vehicle centre of mass (m)

\( b \) distance between the rear axle and the vehicle centre of mass (m)

\( C_y \) product of the cornering stiffnesses of the front and rear tyres (equation (45))

FYNF normalized resultant lateral force of the front tyres = \( F_{yf} \) = \( F_y / C_1 \)

FYNR normalized resultant lateral force of the rear tyres = \( F_{yr} \) = \( F_y / C_2 \)

\( F_{yf} \) resultant lateral force produced by the front tyres

\( F_{yr} \) resultant lateral force produced by the rear tyres

\( F_{ywi} \) tyre lateral force normal to the wheel plane

\( F_{zf} \) combined normal load on the front tyres

\( F_{zr} \) combined normal load on the rear tyres

\( g \) acceleration due to gravity

\( i \) wheel index = 1, 2, 3, 4

\( I_{zz} \) yaw moment of inertia of the vehicle (kg\( \cdot \)m\(^2\))

\( l \) wheelbase of the vehicle (m)

\( m \) mass of the vehicle (kg)

\( r \) yawing velocity

\( R \) radius of curvature

\( u \) forward velocity of the vehicle’s mass centre

\( v \) lateral velocity of the vehicle’s mass centre

\( V \) resultant velocity of the vehicle’s mass centre

\( V_{cr} \) critical speed

\( V^2 / (gR) \) lateral acceleration in terms of unit \( g \)

\( W \) weight of the vehicle (N)

\( \text{YM} \) yawing moment coefficient (equation (44))

\( \alpha \) slip angle of the tyre

\( \alpha_f \) slip angle of the front tyres

\( \alpha_r \) slip angle of the rear tyres

\( \beta \) sideslip angle of the vehicle

\( \delta \) steer angle of the front wheels

\( \delta_y \) resultant lateral stiffness of the front tyres (N/\( \text{rad} \))

\( \delta_z \) resultant lateral stiffness of the rear tyres (N/\( \text{rad} \))