

# Voting Systems

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Decisions are made every day in every organization. The most famous decisions are public elections. Many people assume that the election results always indicate the will of the voters. You might be surprised to find out

that the election results depend as much on the voting system or election method.

## 1 Plurality Voting

The voting system most people are familiar with is called plurality voting. Plurality voting is the voting system used in most state and local elections in the United States.

*Plurality voting: the candidate with the most votes wins.*

To introduce you to more kinds of voting systems, we will examine a vote in which plurality voting might not give what seems to be the fairest vote.

A math class is voting on when they would like to have a test review session. Lydia favors an evening session, so she can go after she gets out of work. An afternoon session would be okay with her, but she cannot get to a morning session. Her voting preference can be written EAM, where E (evening) is her first choice, A (afternoon) her second choice and M (morning) is her last choice. Five other people in the class have the same preference as Lydia. We can show this preference on a chart:

|                   |   |
|-------------------|---|
| number of voters: | 6 |
| first choice      | E |
| second choice     | A |
| third choice      | M |

This means that six people (including Lydia) have the preference EAM.

Other people in the class do not share Lydia's preference for evenings. Ten people in the class would really like a morning review session, but an evening session would be okay. Their preference would be written MEA. Seven people in the class have the preference AEM. We can put all the voting preferences together in the table:

|                   |   |    |   |
|-------------------|---|----|---|
| number of voters: | 6 | 10 | 7 |
| first choice      | E | M  | A |
| second choice     | A | E  | E |
| third choice      | M | A  | M |

By plurality voting, the winner would be the morning session, since M has ten votes. However, if you look at the voting chart, you can see that  $6 + 7 = 13$  people have morning as their *last* choice. You can also see that a good compromise might have been an evening session, since 6 people liked evening best, and  $10 + 7 = 17$  people had evening as their second choice. Under plurality voting, only the students' first place choices were used.

*Plurality voting does not take into account people's second or third place choices.*

Thus, one of the consequences of using plurality voting for some offices is that voters' second and third choices are not used in determining the winner. If the two top candidates are very close, as with the morning and evening choices in our class vote, giving weight to second and third choices might help us reach a compromise that more people would like.

**Try it out:**

In the following election for mayor, who would be the plurality winner, Dan Dirkman (D), Jermael Johnson (J), or Candace Carlin (C)?

|                   |   |   |   |   |   |
|-------------------|---|---|---|---|---|
| number of voters: | 5 | 2 | 3 | 1 | 4 |
| first choice      | J | D | C | J | C |
| second choice     | C | J | D | D | J |
| third choice      | D | C | J | C | D |

*Hint:* count only the first place votes in each preference column. For example, since you see J (Jermael Johnson) in first place in two columns, you should count both those columns.

## 2 Plurality with Elimination

One way to take into account people's second-place preferences is called plurality with elimination, also known as instant runoff voting.

*Plurality with elimination (or instant runoff): the candidate with the least first place votes is eliminated, and his votes are transferred to the next candidate in the ranking.*

Using our previous example of the students voting for a test review time, if Lydia has the preference EAM and E loses, her vote goes to A, her second choice.

|                   |              |              |              |
|-------------------|--------------|--------------|--------------|
| number of voters: | 6            | 10           | 7            |
| first choice      | <del>E</del> | M            | A            |
| second choice     | A            | <del>E</del> | <del>E</del> |
| third choice      | M            | A            | M            |

Now A gets  $6 + 7 = 13$  votes and wins, so the session is held in the afternoon.

Plurality with elimination is used in some elections in Australia and other places. In America, it was used in party primaries in several states in the first half of the 20th Century. However, it fell out of favor and was dropped. Currently, instant runoff is gaining popularity.

We will now look at another example where all possible voting preferences are shown. Suppose the fifteen member student senate wants to elect one of three nominees for the College's Professor of the Year award. The nominees are from the Art (A), Business (B), and Chemistry (C) departments. The preferences for each student senator are summarized below:

|                   |   |   |   |   |   |   |
|-------------------|---|---|---|---|---|---|
| number of voters: | 2 | 5 | 0 | 1 | 4 | 3 |
| first choice      | A | A | C | C | B | B |
| second choice     | B | C | A | B | C | A |
| third choice      | C | B | B | A | A | C |

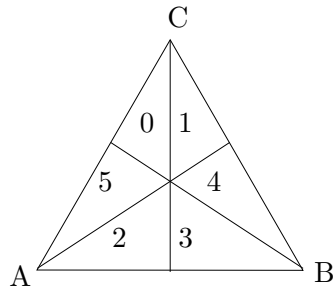
Notice that nobody has the preference CAB.

Using the plurality method and counting the first place votes, we have a tie: A gets 7 votes, B gets 7 votes, and C gets 1 vote.

The senate would like a clear majority for its selection, so it decides to use plurality with elimination. Candidate C is eliminated. The one person whose first choice was C now votes for B. Candidate B gets  $1 + 4 + 3 = 8$  votes and wins with a majority of the votes against 7 for A.

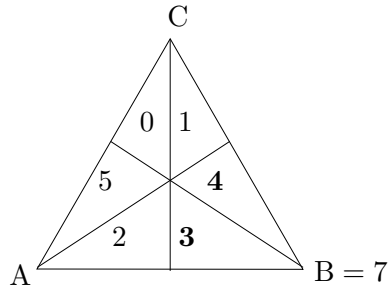
When there are many voting preferences, another way to illustrate them is called a *Saari representation triangle* (after Donald G. Saari, who created them in the 1990's). The triangle can make tallying total votes much easier, especially when other voting systems are used. *The representation triangle shows all possible voter preferences.*

We begin with a triangle with our three choices, one on each vertex, and lines drawn from each vertex to the middle of the opposite side. In each region, we put the number of votes for each voting preference:



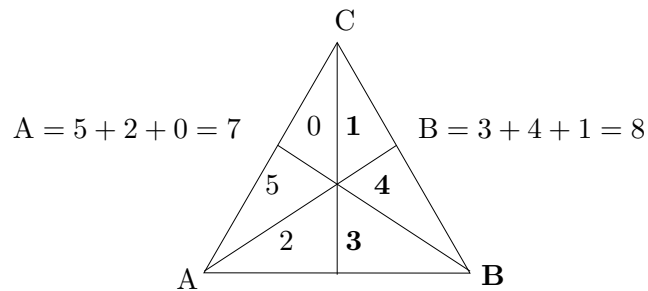
To tell which region corresponds to each voting preference, look at which vertex is closest to that region. For example, the region with 4 votes is closest to vertex B. Vertex C is the next closest and vertex A is farthest away, so the 4 corresponds to the preference BCA. Without looking at the table, try to figure out which voter preference corresponds to the number 5. Then check the table to see if you are correct.

Using the Saari triangle, we can get the *plurality totals* by counting the two regions closest to each vertex. For example, the total plurality vote for B is:



Verify that the total plurality vote for A is also 7, using the correct regions closest to A. What is the plurality total for C?

We can also use the triangle to help us find the total plurality with elimination tallies. Since C is eliminated, the tallies become the totals on the A side and the B side of the triangle:



*In the Saari triangle, the plurality voting totals can be obtained by adding the two regions closest to each vertex. Plurality with elimination totals can be gotten by adding the three regions on each side of the two candidates left after eliminating the losing candidate.*

**Try it out:**

A group of 12 friends is trying to decide what type of food to have for dinner. Their choices of cuisine are: American (A), Brazilian (B) and Chinese (C). The group’s preferences are summarized in the table below:

|               |   |   |   |   |   |   |
|---------------|---|---|---|---|---|---|
|               | 2 | 3 | 3 | 2 | 2 | 0 |
| first choice  | A | A | C | C | B | B |
| second choice | B | C | A | B | C | A |
| third choice  | C | B | B | A | A | C |

Use the table to create a preference triangle. Then use the triangle to determine the plurality and plurality with elimination winners.

### 3 Sequential Pairwise Voting

*Sequential Pairwise voting: choose two candidates for a head-to-head contest. The winner moves on to face a third candidate in a head-to-head contest, with the winner of that contest moving on to face a fourth candidate. Continue until only one candidate is left.*

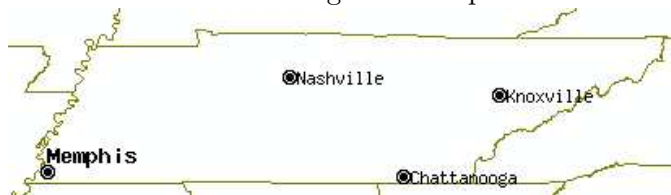
With three or more candidates, a sequential pairwise election can occur in several ways, depending on the order in which candidates are to face-off.

**Example:** (Source: Wikipedia.org)

Imagine an election for the capital of Tennessee (Figure 1). Let’s say the candidates for the capital are Memphis (M), Nashville (N) and Knoxville (K). Here’s the population breakdown by metro area (surrounding county):

|                              |         |
|------------------------------|---------|
| Memphis (Shelby County):     | 826,330 |
| Nashville (Davidson County): | 510,784 |
| Knoxville (Knox County):     | 621,285 |

Figure 1: Map of Tennessee



Suppose the voters vote based on geographic proximity. In other words, voters prefer the capital to be as close as possible to them. If we assume that everyone in the state lives in the three metro areas, then the preferences will be as follows:

|                    |     |     |     |
|--------------------|-----|-----|-----|
| Percent of voters: | 42% | 26% | 32% |
| first choice       | M   | N   | K   |
| second choice      | N   | K   | N   |
| third choice       | K   | M   | M   |

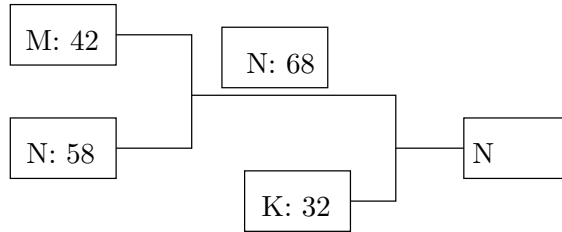
**Three examples of how to calculate the votes in different lists:**

1. For the order M vs. N, and the winner vs. K:

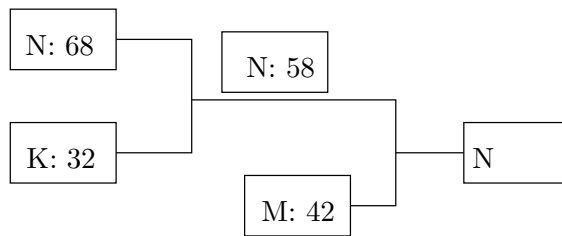
To find the winner between M and N, we look at how many total voters preferred M over N and N over M. In the first column 42% of the voters preferred M over N, in the second column 26% of the voters preferred N over M and in the third column 32% of the voters preferred N over M. Now the total number of voters who preferred M over N = 42% and the total number of voters who preferred N over M  $26 + 32 = 58\%$ . Hence N wins.

Next we pair N (the winner of the first round) with K and do the process again for these two. In the 1st column 42% preferred N over K, in the 2nd column 26% preferred N over K and in the 3rd column 32% preferred K over N. Now the total number of voters who preferred N over K  $42 + 26 = 68\%$  and the total number of voters who preferred K over N = 32%. Hence N wins.

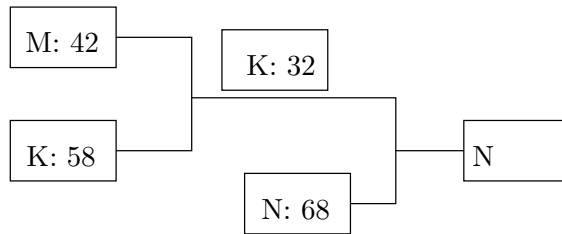
The following diagram illustrates the results of this pairwise process.



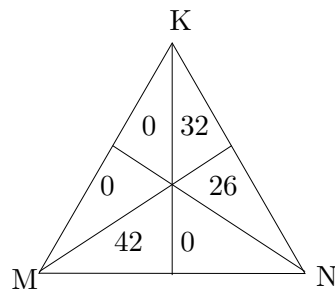
2. For the order N vs. K, and the winner vs. M:



3. For the order M vs. K, and the winner vs. N:



Using the Saari Triangle:



1. For the head-to-head contest M vs. N:

In order to find the winner between M and N, we first look at the three regions representing rankings that prefer M over N: MNK (42), MKN (0) and KMN (0).

Thus, the total (42) of the votes for M over N can be obtained from the triangle by adding the numbers in these three regions-the regions to the *left* of the vertical line.

Now we look at the regions representing rankings that prefer N over M: KNM (32), NKM (26) and NMK (0).

Notice that the total (58) of the votes for N over M can be found by adding the numbers in these three regions-the regions to the *right* of the vertical line.

In summary:

M vs N: 42 vs 58. N is the winner.

2. For the contest N vs. K:

Using this approach for the N vs. K vote, the N-votes are on the half of the triangle closest to the N vertex:  $N = 42 + 0 + 26 = 68$ . The K-votes are on the half-triangle closest to the K vertex:  $K = 0 + 0 + 32$ .

N vs. K: 68 vs. 32. N is the winner.

3. Similarly we could find the winner for the pairing (M, K).

**Try it out:**

In a survey, 8 people were asked to rank Pepsi (P), Cola (C) and Dr. Pepper (D) according to their preferences. The following is the distribution of their rankings.

| 3 voters | 3 voters | 2 voters |
|----------|----------|----------|
| P        | D        | C        |
| C        | P        | D        |
| D        | C        | P        |

1. Start with the pair (P, C); the winner vs. D.

Winner:

2. Start with the pair (P, D); the winner vs. C.

Winner:

3. Start with the pair (D, C); the winner vs. P.

Winner:

Is there a clear winner? Explain.

## 4 Condorcet's method of pairwise comparison

The only “perfect” election (one that truly reflects the desires of the population) occurs when there are two candidates. The winner is the candidate who receives a majority of the votes and there is no doubt who beat whom. The 18th-century French mathematician and philosopher Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet, proposed a method which uses the idea of a two-candidate election to determine the winner when there are three or more candidates. His idea reduces down to looking at a series of head-to-head matchups between the candidates where the overall winner is the candidate who wins the majority of these sub-contests. To allow for tied matchups, we will give one point to a candidate who wins a matchup outright and 1/2-point to each candidate if there is a tie. The winner then will be the person with the most points.

*In the Method of Pairwise Comparison, each candidate is matched with every other candidate in a one-on-one election. The winner of this sub-contest earns one point and if there is a tie, both candidates receive 1/2 point. The winner is the candidate with the most points.*

**Example:**

Start with the preference table below for an election of Dogcatcher for Catscratch County, PA. Here we have three candidates: Smith, Jones, and Firkaly.

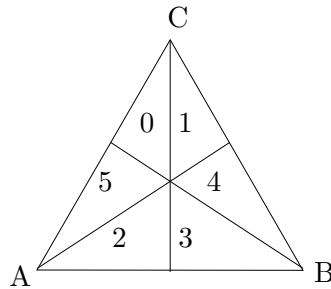
|                   |         |         |         |         |
|-------------------|---------|---------|---------|---------|
| number of voters: | 20      | 10      | 9       | 8       |
| first choice      | Smith   | Firkaly | Firkaly | Jones   |
| second choice     | Firkaly | Smith   | Jones   | Firkaly |
| third choice      | Jones   | Jones   | Smith   | Smith   |

We will have three head-to-head matchups: Smith v. Jones, Smith v. Firkaly, and Jones v. Firkaly. Let us look at the first one. In this instance, the third candidate (Firkaly) is ignored and the chart looks like this:

|                   |       |       |       |       |
|-------------------|-------|-------|-------|-------|
| number of voters: | 20    | 10    | 9     | 8     |
| first choice      | Smith |       |       | Jones |
| second choice     |       | Smith | Jones |       |
| third choice      | Jones | Jones | Smith | Smith |

Now Smith is above Jones in the first two columns only, giving 30 total votes for Smith while 17 voters preferred Jones. Thus Smith gets 1 point. Similarly, Firkaly beats Smith (27 to 20), and Firkaly beats Jones (39 to 8). So Firkaly has 2 points, Smith 1, and Jones 0. Notice that Firkaly won the election without having a plurality of first-place votes.

We can use this method with the Saari triangles already introduced. In the figure below, the ballots which rank Candidate A above Candidate B are those that abut the corner marked A. Similarly, those who prefer B to A are the three triangles which touch the B corner. Thus it is  $0 + 5 + 2 = 7$  votes for A, and  $1 + 4 + 3 = 8$  votes for B and B earns a point. Continuing onward, A defeats C and B defeats C, giving B the overall win.



The simplicity of this idea leads to what is now called the Condorcet Criterion for a fair election. Basically if the winner of an election is anyone other than a Condorcet winner, then the election must be flawed. d

*Condorcet Criterion: For an election to be fair, the winner of the election should be the candidate who wins the most of the pairwise elections.*

For a real example of this, we turn to the 2000 Presidential Election between Al Gore, George W. Bush, and Ralph Nader. All data indicates that if it was a two-candidate election Gore would have beaten Bush and Gore would have won against Nader. So Bush was the electoral winner, but a Condorcet loser.

Are there drawbacks to the system? Yes, there are three worth commenting on.

First, this method can degenerate into a cycle. An example of one is below.

|                   |    |    |   |
|-------------------|----|----|---|
| number of voters: | 12 | 10 | 9 |
| first choice      | A  | B  | C |
| second choice     | C  | A  | B |
| third choice      | B  | C  | A |

In A against B, B wins; in B against C, C wins; and in C against A, A wins. Thus all candidates tie and no winner is found. This can easily be overcome by putting in a contingency plan for how to break a cycle (e.g. using plurality to determine the winner among the candidates in the cycle).

The second problem is not so easily solved since it lies not with the system, but with the voters. Condorcet methods are vulnerable to what is called “burying.” In burying a voter places a candidate lower than he/she sincerely feels in hopes of making a more preferred candidate able to have more head-to-head wins than the falsely lowered one.

Lastly, some experts in voting theory believe that Condorcet methods discourage candidates from revealing hard-line stances on some issues. This is due to the fact that the least offensive candidate will end up being ranked higher than the hard-liners and thus be able to win. The result of this is a less well-informed electorate.

Examples of variations on the Condorcet method include the Black Method (named after Duncan Black), the Minimax method, and the Schulze Method. The Software in the Public Interest Corporation uses Schulze to elect member of its board of directors as does the Debian Project (which runs a free computer operating system service).

## 5 Borda Count

*Borda count: each voter submits his/her voting preferences ranking of the alternatives to be considered. The candidate in the last place receives 0 points; the next-to-last place 1 point; the next highest receives 2 points and so on. The number of points for each alternate is then totaled and the candidate with highest total is the winner.*

For example, in the case of three candidates, a voter's first choice will receive 2 points, 2nd choice 1 point and the last choice 0 point. In this method the candidates that are not a voter's first choice are not totally neglected as they are in the plurality method.

### **Example:**

The 1912 U. S. Presidential race was between Theodore Roosevelt (Bull Moose Party), William Taft (Democrat) and Woodrow Wilson (Republican). In the election, Wilson received about 45% of the vote, Roosevelt received about 30% of the vote and Taft received about 25% of the vote. Wilson won the Presidency. According to political historians, voters for Wilson generally preferred Roosevelt over Taft. Roosevelt voters supported Taft over Wilson and Taft voters supported Roosevelt instead of Wilson.

| percent of voters: | 45% | 30% | 25% |
|--------------------|-----|-----|-----|
| first choice       | W   | R   | T   |
| second choice      | R   | T   | R   |
| third choice       | T   | W   | W   |

Determining the winner using the Borda count:

Wilson received 45% of the votes as the first choice candidate and 55% as last choice candidate. His Borda count is

$$W = 45(2) + 0(1) + 55(0) = 90.$$

Roosevelt received 30% of the votes as the first choice candidate, 70% as second choice candidate and 0% as last choice candidate, so his Borda count is

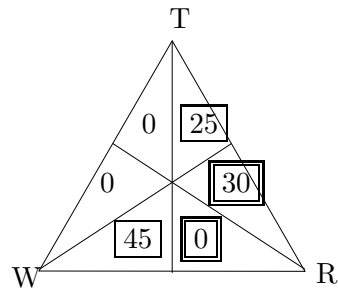
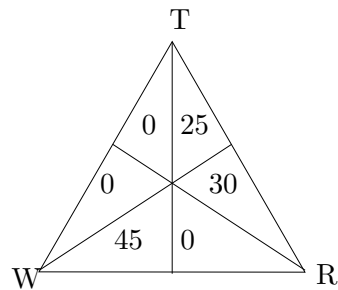
$$R = 30(2) + 70(1) + 0(0) = 130.$$

Likewise, Taft received 25% of the votes as the first choice candidate, 30% as second choice candidate and 45% as last choice candidate. His Borda count is therefore

$$T = 25(2) + 30(1) + 45(0) = 80.$$

**Conclusion:** Roosevelt wins using the Borda count method.

In the case of three candidates one can also use the Saari triangle to calculate the Borda count. For the above example, we get the following triangles:



Let's compute the Borda count for each candidate beginning with Roosevelt (R).

From the triangle above the numbers enclosed by double boxes indicate Roosevelt as first choice (2 points each vote), the ones enclosed by single boxes as second choice (1 point each vote) and the unboxed numbers as last choice (0 points each) candidate. Thus the Borda count for Roosevelt is

$$R = (30 + 0)(2) + (25 + 45)(1) + (0 + 0)(0) = 130.$$

Similarly, the Borda count for Wilson is

$$W = (45 + 0)(2) + (0 + 0)(1) + (25 + 30)(0) = 90,$$

and for Taft is

$$T = (25 + 0)(2) + (30 + 0)(1) + (45 + 0)(0) = 80.$$

Under the Borda count Roosevelt is the winner while Wilson won the plurality vote (and the presidency).

**Try it out:**

In a survey, 100 people were asked to rank Pepsi (P), Coke (C) and Dr. Pepper (D) according to their preferences. The following is the distribution of their rankings. Determine which beverage people prefer using the Borda count first by hand and then using Saari triangle.

|                   |    |    |    |   |
|-------------------|----|----|----|---|
| number of voters: | 32 | 35 | 25 | 8 |
| first choice      | P  | C  | D  | C |
| second choice     | D  | P  | C  | D |
| third choice      | C  | D  | P  | P |

## 6 Desirable properties in voting systems

In the previous sections we discussed several different voting systems and found that the results depend on the election method, confusing what could be called the will of the voters. Since this can happen we begin to wonder about our methods and then question arises "How do we evaluate the voting system under consideration?" To answer this, we first need to make up how we want it to be. In other words, what are the desirable properties of voting systems?

In two party system (two alternatives) the plurality (which is really the majority of votes) method is most commonly used method as it treats all

the votes equally. This is the best system for choosing the winner in that situation. But what if we have more than two alternatives? As we saw in previous section situation gets complicated if there are more than two choices.

With three or more alternatives (multi-party system) an individual can be the plurality winner even though he/she may get less than half of the total population vote. In this case many of the voters may not be satisfied with the decision. An example is the 2000 U.S. Presidential election in which there were three major candidates; George W. Bush, Al Gore, and Ralph Nader. Bush won the election although he did not get the majority of the votes. For more information see the web site <http://www.uselectionatlas.org/>. Satisfying a majority will be a part of one of our criteria.

If we agree that plurality does not satisfy many of the voters, then one can think of using the Borda count as it gives the weight to the lower rankings. This system also has a drawback. Consider an election among three candidates (A, B, C) where A is ranked over B, yet when the election is done again and not a single voter switched the order of the preference for A and B, B now wins. This is shown in the example below.

Prior to an election a (very small) poll is taken. It has preference table

|                   |   |   |   |
|-------------------|---|---|---|
| number of voters: | 5 | 3 | 3 |
| first choice      | A | B | C |
| second choice     | C | A | B |
| third choice      | B | C | A |

Using Borda Count candidate A has 24 points to B's 20 (and C's 22). Then, in a second poll right before election day, the five voters in the first column reverse the order of B and C (thus preserving A over B). The table now reads

|                   |   |   |   |
|-------------------|---|---|---|
| number of voters: | 5 | 3 | 3 |
| first choice      | A | B | C |
| second choice     | B | A | B |
| third choice      | C | C | A |

Now A still has 24 points, but B has 25 (not only beating A, but this time winning the election).

So it seems that a voting system should have the property of picking the same winner independent of what happens to candidates ranked below the winner. This will, in turn, disallow strategic nominations of the candidates. Again, this will be a part of one of our properties.

The following are the most commonly considered criteria for a “satisfying” election:

Non-dictatorship: The decision-making should not simply follow the preference order of a single individual while ignoring all others.

Majority: If a majority of voters strictly prefers a given candidate to every other candidate and the voting is sincere, then that candidate should win.

Monotonicity: If an alternative X loses, and, in a second election, the only change to ballots is placing X in lower positions, then X must still lose. (It is generally considered a good thing if a voting system is monotonic since non-monotonic systems are vulnerable to tactical voting, where voters might try to elect their candidate by purposely lowering that candidate.)

Condorcet criterion: If a choice beats the majority of other choices in pairwise comparison, it should win.

Independence of irrelevant alternatives (I.I.A): If A is preferred to B in an election, then introducing a third, irrelevant, alternative X into a re-election should not make B preferred to A. In other words, whether A or B is better should not be changed by the availability of X.

Consistency: If the electorate is divided in two and choice A wins in both parts, then choice A should win the overall election.

Besides these, there are more extensive lists of criteria on the voting systems. We refer the interested reader to “*Electoral Systems: A Comparative Introduction*,” by David M. Farrell for further information.

Now we look at those voting systems that we discussed in previous section satisfy above properties.

|                  | Plurality | Instant Runoff | Pairwise Comparison | Borda Count |
|------------------|-----------|----------------|---------------------|-------------|
| Non-dictatorship | P         | P              | P                   | P           |
| Majority         | P         | P              | P                   | <b>F</b>    |
| Monotonicity     | P         | <b>F</b>       | P                   | P           |
| Condorcet        | <b>F</b>  | <b>F</b>       | P                   | <b>F</b>    |
| I.I.A.           | <b>F</b>  | <b>F</b>       | <b>F</b>            | <b>F</b>    |
| Consistency      | P         | <b>F</b>       | <b>F</b>            | P           |

P: pass; **F**: fail

For the interested reader, some other desirable properties of voting system include simplicity, ambiguity of the method, speed of counting ballots, reduction for fraud and dispute.

As we see in the above table, no voting system is perfect. This can be discouraging and raises the new question, “Is there a voting system that satisfies all the desirable properties indicated above?” It is known that it is impossible for one voting system to pass all criteria. One such result, determined by Kenneth Arrow and called Arrow’s Impossibility Theorem, showed that non-dictatorship, monotonicity, and independence of irrelevant alternatives are mutually contradictory for ranked voting systems. This means we cannot make our systems perfect. Now he is not saying that these criteria will always be violated, but that examples will exist (and can exist in real life) which violate these ideas for a “fair” election. For this Arrow won a 1972 Nobel Prize.

Some argue that Arrow required strong assumptions about what makes a voting method fair, but there is no inherent reason that his criteria should be considered fair. For example, Arrow requires implicitly that the voting method uses a ranked ballot, but it would be narrow-minded to state that any voting method that does not use a ranked ballot is unfair. If a method requires a random decision to break a tie, Arrow’s criteria bring the decision down to an unfair one. They also point out that the independence of irrelevant alternatives (I.I.A.) is a flawed criterion. For example, if a population slightly preferred candidate B to candidate A, but candidate A supporters were far more loyal, then an introduction of a third candidate could split B’s support far more than A’s, leading to a win by A. In cases where one candidate’s supporters feel they are compromising far more than the other candidate’s supporters do, failing I.I.A. may not be a flaw.

Despite all these argument on Arrow’s theorem, it is still considered useful in the sense that it provided a mathematical way of looking at the voting systems and their limitations.

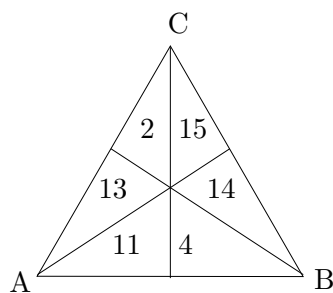
In conclusion, someone implementing a voting system has to decide which criteria are important for the election. We should not forget that voting systems are not only just theoretical models but are implemented in real life. So one should also keep in mind while choosing the voting system the practicality of the system.

## 7 Exercises and Activities

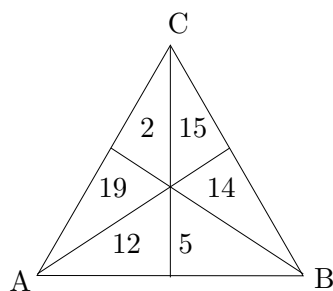
### Exercises:

- In Plurality with Elimination, there is more than one way to decide how to eliminate a candidate. For example, you can eliminate the person with the *least first place votes*, or you can eliminate the candidate with the *most last place votes*. For each of the Saari representation triangles below, determine (i) the Plurality winner, (ii) the Plurality with Elimination winner when eliminating by least first place votes, (iii) the Plurality with Elimination winner when eliminating by most last place votes, (iv) the Condorcet winner, and (v) the Borda count winner.

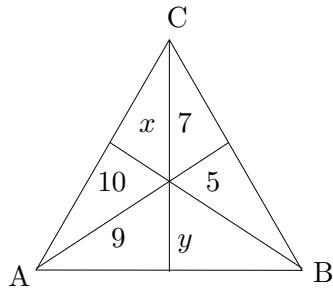
(a) For the triangle



(b) For the triangle



- For the triangle below, determine the values of  $x$  and  $y$  so that simultaneously (1)  $A$  is the plurality winner and (2)  $B$  wins plurality with elimination. Who wins your Borda count?



3. Determine the Condorcet winner from the following preference table:

|                   |   |   |   |   |
|-------------------|---|---|---|---|
| number of voters: | 5 | 4 | 4 | 3 |
| first choice      | A | B | C | C |
| second choice     | B | C | B | A |
| thrid choice      | C | A | A | B |

4. Given the table, construct a preference ballot to create a Condorcet cycle.

|                   |    |   |   |
|-------------------|----|---|---|
| number of voters: | 10 | 9 | 6 |
| first choice      | A  | B |   |
| second choice     | B  | C |   |
| thrid choice      | C  | A |   |

5. How many head-to-head matchups are there for 4 candidates? 5?  $N$ ?
6. Explain why this table cannot lead to a Condorcet cycle regardless of the order in the last column.

|                   |   |   |   |
|-------------------|---|---|---|
| number of voters: | 7 | 6 | 5 |
| first choice      | A | B |   |
| second choice     | B | A |   |
| thrid choice      | C | C |   |

7. Must a candidate who wins a *majority* of the first place votes be the Condorcet winner? Explain your answer.
8. The selection of Major League Baseball's (MLB) American League (AL) Most Valuable Player is done via a Borda count where two sportswriters from each city with an AL team make up the voters. For the 1999 season the top five candidates were as below.

| Player       | Team    | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|--------------|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
| I. Rodriguez | Rangers | 7   | 6   | 7   | 0   | 5   | 2   | 1   | 0   | 0   | 0    |
| P. Martinez  | Red Sox | 8   | 6   | 4   | 1   | 2   | 2   | 3   | 0   | 0   | 0    |
| R. Alomar    | Indians | 4   | 7   | 6   | 4   | 4   | 0   | 1   | 1   | 0   | 0    |
| M. Ramirez   | Indians | 4   | 4   | 5   | 9   | 1   | 5   | 0   | 0   | 0   | 0    |
| R. Palmeiro  | Rangers | 4   | 1   | 2   | 4   | 8   | 4   | 5   | 1   | 0   | 1    |

- (a) A variation of the Borda count is used by MLB, giving 14 points for first place, 9 for second, 8 for third, . . . , 1 point for last. Find the rankings of these five players using this method.
- (b) Find the rankings of these five players using the usual Borda count.
- (c) Find the rankings if we only look at the first three players and their first, second, and third place votes.
- (d) Find the rankings if we only look at the first four players and their first, second, third, and fourth place votes.
- (e) Find the rankings for all five players if we only use their first through fifth place votes.
9. During a classroom evaluation, certain aspects are rated A = excellent through E = horrible. To find an “average,” a university assigns points to each level: A = 4, B = 3, C = 2, D = 1, and E = 0. Dividing the Borda total by the number of responses gives this average. For example,

The manner in which this class was taught was \_\_\_\_\_.

|          |    |    |   |   |   |
|----------|----|----|---|---|---|
| response | A  | B  | C | D | E |
| number   | 20 | 10 | 5 | 2 | 3 |

The average for this question would be

$$\frac{20(4) + 10(3) + 5(2) + 3(0)}{20 + 10 + 5 + 2 + 3} = \frac{122}{40} = 3.05.$$

- (a) One professor decided to make her own averages using A = 5, B = 3, C = 2, D = 1, and E = 0. Why will this enable her to look better than she really is in the eyes of the administration?

- (b) Faculty complain that  $E = 0$  is unfair because multiplying by zero is just another zero. If the administration agrees and changes to  $A = 5$ ,  $B = 4$ ,  $C = 3$ ,  $D = 2$ , and  $E = 1$ , by how much will that raise the scores? Does it really make a difference?
10. The table below shows an extrapolation of the data from the 1991 Louisiana Gubernatorial race, pitting moderate Republican incumbent Buddy Roemer against Democrat Edwin Edwards and conservative Republican David Duke. In this problem, other candidates are ignored.

|                      |    |    |   |    |   |
|----------------------|----|----|---|----|---|
| Voter Group:         | 1  | 2  | 3 | 4  | 5 |
| Preferences:         | E  | R  | R | D  | D |
|                      | R  | E  | D | R  | E |
|                      | D  | D  | E | E  | R |
| Proportion of voters | 34 | 23 | 4 | 30 | 2 |

- (a) Who would have won a plurality contest?
- (b) Who would have been the Borda count winner?
- (c) What results would have been reached if we used a plurality with elimination of the last place candidates?
- (d) A traditional primary method would have pitted Roemer against Duke (both being Republicans), with the winner going against Edwards. Who would have won in this system? (A traditional primary method is not used in Louisiana.)
- (e) In your opinion, who should have won and why?
11. Consider the following preferences on four competing bills (A, B, C, and D) in the U.S. Senate:

|                   |    |    |    |
|-------------------|----|----|----|
| Voter group:      | 1  | 2  | 3  |
| Preferences:      | A  | B  | C  |
|                   | B  | C  | D  |
|                   | C  | D  | A  |
|                   | D  | A  | B  |
| Number of voters: | 40 | 32 | 28 |

- (a) In your opinion, which bill should be passed? Why?
- (b) Create a sequential pairwise agenda that contains all candidates and produces bill D as the winner.

**Activity: Conduct a classroom election.**

Have everyone in your class rank their preference of Coca Cola, Pepsi and Dr. Pepper.

- a. Before looking at the data, decide on an election method to be used.
- b. Show the class data in a Saari representation triangle.
- c. Analyze the election using your election method.
- d. Analyze the election using the other election methods you read about. Are there differences?

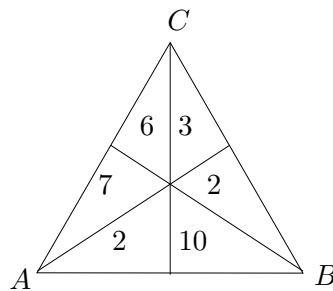
**Activity: Does my vote count?**

For this part of the module you will explore three different situations relatively common during elections. They refer to cases when the voters decide not to cast their votes. Using the Saari representation triangle, you will compare the results for the plurality, Borda count, and pairwise voting. Remember that when we write  $ABC$  it means that candidate  $A$  is the first choice,  $B$  is the second choice, and  $C$  is the third choice. These are the three common situations to consider:

1. Your parents have different preferences to vote for candidates  $A$ ,  $B$ , and  $C$ . Both have candidate  $C$  as the second choice, but they disagree on the first and last choice. After discussing their preferences they decide to refrain from voting.
2. Your parents disagree on the first and second choices, but agree on the last choice.
3. Your father believes his vote will not affect the result of the election and for that reason he decides not to vote.

**SITUATION 0**

The preferences of your parents are included in the tally given in the Saari representation triangle below.

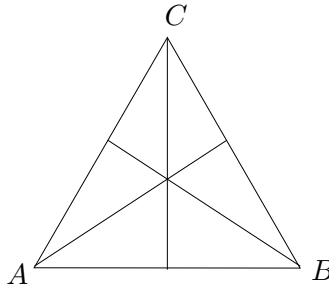


1. Determine the preference of the voters using the plurality vote.
2. Determine the preference of the voters using the Borda count vote.
3. Who would win each of the different pairwise races?

SITUATION 1

Your Mom and Dad have  $ACB$  and  $BCA$ , respectively, as their preferences. Since they have the same second choice but different first choices, they decided not to vote.

1. Record the new tally in the triangle below:



2. Refer to the triangle to complete the following table:

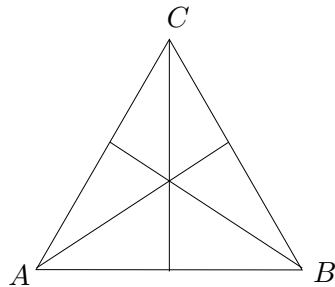
| Election Results | Plurality | Borda Count | Pairwise    |             |             |
|------------------|-----------|-------------|-------------|-------------|-------------|
|                  |           |             | $A$ vs. $B$ | $A$ vs. $C$ | $B$ vs. $C$ |
| SITUATION 0      |           |             |             |             |             |
| SITUATION 1      |           |             |             |             |             |

3. Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.
4. Under which voting system does candidate  $C$  benefit from your parents not voting?
5. Other couples, friends of your parents, have the same preferences as your parents. If you convince them not to vote, in which of the three systems would candidate  $C$  win? How many couples do you need to convince not to vote?

SITUATION 2

Your Mom and Dad now have as preferences  $ABC$  and  $BAC$  respectively. Once again they feel their votes will not contribute to the final outcome, and therefore they do not vote.

- Record the new tally in the triangle below:



- Refer to the triangle to complete the following table:

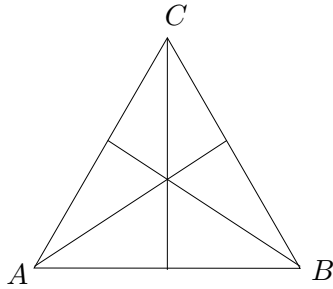
| Election Results | Plurality | Borda Count | Pairwise    |             |             |
|------------------|-----------|-------------|-------------|-------------|-------------|
|                  |           |             | $A$ vs. $B$ | $A$ vs. $C$ | $B$ vs. $C$ |
| SITUATION 0      |           |             |             |             |             |
| SITUATION 1      |           |             |             |             |             |

- Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.
- Under which voting system does candidate  $C$  benefit from your parents not voting? (Plurality, Borda, pairwise)
- Other couples, friends of your parents, have the same preferences as your parents. If you convince them not to vote, in which of the three systems would candidate  $C$  win? How many couples do you need to convince not to vote?

SITUATION 3

Your father does not feel like voting since his candidate is “ahead” and a sure winner. His preference is  $BAC$ . He is convinced his vote will not change the final outcome.

- Record the new tally in the triangle below:



2. Refer to the triangle to complete the following table:

| Election Results | Plurality | Borda Count | Pairwise       |                |                |
|------------------|-----------|-------------|----------------|----------------|----------------|
|                  |           |             | <i>A vs. B</i> | <i>A vs. C</i> | <i>B vs. C</i> |
| SITUATION 0      |           |             |                |                |                |
| SITUATION 1      |           |             |                |                |                |

3. Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.
4. Your father has friends with the same preferences as he. If three such friends refrain from voting, candidate *B* will lose the plurality voting. How many voters can refrain from voting so that he can still win using another voting technique? Explain.

Finally, write a paragraph to summarize your conclusions about the results you just obtained dealing with the three different situations.