BMO_α^p SPACES AND THEIR PROPERTIES

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Abstract. We introduce the function space BMO_α^p, which generalizes both BMO space of Bergman metric and α-Bloch space. By establishing certain covering property for the open unit disc, we prove that a BMO_α^p function can be decomposed into a "bad" function which satisfies a Carleson type condition and a "good" function which satisfies a Sobolev type condition. As an application we also characterize the multipliers from BMO_α^p to BMO_α^q and VMO_α^p to VMO_α^q spaces respectively.

1. **Introduction.**

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, and let dA be the normalized area measure. The BMO space of Bergamn metric on the open unit disc was first introduced in [2]. It is the collection of all functions F on $\mathbb{D}$ such that (see Zhu's book [11])

$$\sup_{z\in\mathbb{D}} \inf_{c} \frac{1}{(1-|z|)^2} \int_{|w-z|<\frac{1}{2}(1-|z|)} |F(w)-c|^2 \, dA(w) < \infty.$$

BMO space plays an important role in modern analysis, especially in operator theory and function spaces. We refer the reader to [3], [4], [5], [6] [9], [10], [11]and references therein. The α-Bloch space is the collection of all harmonic functions F on $\mathbb{D}$ such that $(1-|z|)^\alpha |\nabla F(z)|$ or equivalently

$$\frac{1}{(1-|z|)^{2\alpha}} \int_{|w-z|<\frac{1}{2}(1-|z|)^\alpha} |F(w)-F(z)| \, dA(w)$$

is bounded on $\mathbb{D}$. For the properties of this space we refere the reader to [7], [8] and [1] and references therein.

In this paper we study a class of spaces BMO_α^p where $\alpha > 1$ and $1 \le p < \infty$ which generalizes both BMO and α-Bloch spaces.

For fixed $t \in (0,1)$, $\alpha \ge 1$ and $z \in \mathbb{D}$, we define the disc $B_\alpha(z,t)$ as follows

$$B_\alpha(z,t) = \{w \in \mathbb{C} : |z-w| < t(1-|z|)^\alpha\}.$$

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Note that $\alpha \geq 1$ is necessary since the disc $B_\alpha(z,t)$ may not remain inside $\mathbb{D}$ if $\alpha < 1$ and $|z| \to 1^-$. Denote by $\widehat{F}(z)$, the mean of F on the disc $B_\alpha(z,t)$, *i.e.*

$$\widehat{F}(z) = \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} F(w)dA(w)$$

where $|B_\alpha(z,t)| = \int_{B_\alpha(z,t)} dA(w) = t^2(1-|z|)^{2\alpha}$ is the area of $B_\alpha(z,t)$. The p-mean oscillation of F on $B_\alpha(z,t)$, for $1 \leq p < \infty$, is defined by

$$MO^p_\alpha(F)(z,t) = \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|F(w) - \widehat{F}(z)\right|^p dA(w)\right)^{\frac{1}{p}} .$$

For $F \in L^1(\mathbb{D})$, we say that F is in BMO^p_α if $MO^p_\alpha(F)(z,t)$, as a function of z, is bounded on $\mathbb{D}$. We say that F is in VMO^p_α if $MO^p_\alpha(F)(z,t) \to 0$ as $|z| \to 1$. By Holder inequality it follows that $BMO^q_\alpha \subseteq BMO^p_\alpha$ if $0 < p \leq q \leq \infty$. It can be seen that the spaces BMO^p_α and VMO^p_α actually do not depend on "t". Therefore we assume that $t \in (0,1)$ is fixed and use the notation $MO^p_\alpha(F)(z)$ instead of $MO^p_\alpha(F)(z,t)$. These spaces, as far as we know, appear to be new in the litrature.

We remark that $BMO^p_\alpha = BMO$ if $\alpha = 1$ and $p = 2$. BMO^p_α generalizes also the α-Bloch space, because $F(z) = \widehat{F}(z)$ if F is harmonic on $\mathbb{D}$. We note that $BMO^p_\alpha \neq BMO^q_\alpha$ if $p \neq q$. We also note that BMO^p_α is a module, since $MO^p_\alpha(F-c) = MO^p_\alpha(F)$ for all $F \in BMO^p_\alpha$ and any constant c. Therefore, to make the multiplier problem meaningful we need to consider a modified norm for BMO^p_α. This will be addressed in section 4. We also remark that the decomposition of a BMO^p_α function into "good" and "bad" parts has it's roots in [3], and the multiplier problem for BMO is studied systematically in [6].

In section 2, we establish some covering properties for $\mathbb{D}$. In section 3, we prove that a BMO^p_α function can be decomposed into a "bad" function that is bounded in terms of Carleson measure norm and a "good" function that is in the Sobolev type space. Sections 4 and 5 are devoted to establish more properties of BMO^p_α and to characterize the multipliers from BMO^p_α to BMO^q_α and VMO^p_α to VMO^q_α respectively.

Throughout this paper p is a fixed number such that $1 \leq p < \infty$, and the letter "C" denotes the positive constant which may vary at each occurance but is independent of the essential variables and quantities. The notation $\asymp$ means compareable, with the constants independat of the functions and quantities involved.

2. A Covering Property.

Recall that for fixed $t \in (0,1)$ and $\alpha > 1$

$$B_\alpha(w,t) = \{z : |z-w| < t(1-|w|)^\alpha\}.$$

By (t,α)-lattice we mean a family of points $\{z_j\}$ in $\mathbb{D}$ such that the collection of discs $\{B_\alpha(z_j, t/3)\}$ is an open cover for $\mathbb{D}$ and that

$$B_\alpha(z_j, t/16) \cap B_\alpha(z_k, t/16) = \emptyset \ , \quad for \ \ j \neq k.$$

Such a lattice can be constructed as follows. Let

$$E_n = \{z \in \mathbb{D} : |z| \leq r_n = 1 - \frac{1}{n+1} \ , \ n \ \geq \ 1\}.$$

Clearly $E_n \subseteq \mathbb{D}$ for all n. Let $n = 1$. Then $\{B_\alpha(w, t/8)\}_{w \in E_1}$ is an open cover for E_1. Since E_1 is compact, the open cover will have a finite subcover, *i.e.* there exist $w_1^{(1)}, ..., w_{k_1}^{(1)}$ in E_1 such that

$$E_1 \subseteq \cup_{j=1}^{k_1} B_\alpha(w_j^{(1)}, t/8).$$

Let $m > 1$ be such that

$$\cup_{j=1}^{k_1} B_\alpha(w_j^{(1)}, t/8) \subset E_m.$$

Consider

$$\mathcal{E}_m = E_m \backslash \left(\cup_{j=1}^{k_1} B_\alpha(w_j^{(1)}, t/8) \right).$$

Then $\mathcal{E}_m$ is compact and hence the open cover $\{B_\alpha(w, t/8)\}_{w \in \mathcal{E}_m}$ has a finite subcover, *i.e.* there exist, say, $w_1^{(m)}, ..., w_{k_2}^{(m)}$ in $\mathcal{E}_m$ such that

$$\mathcal{E}_m \subseteq \cup_{j=1}^{k_2} B_\alpha(w_j^{(m)}, t/8).$$

Clearly $w_1^{(1)}, ..., w_{k_1}^{(1)}, w_1^{(m)}, ..., w_{k_2}^{(m)}$ are points in E_m and

$$E_m \subseteq \left(\cup_{j=1}^{k_1} B_\alpha(w_j^{(2)}, t/8) \right) \cup \left(\cup_{j=1}^{k_2} B_\alpha(w_j^{(m)}, t/8) \right).$$

Therefore the collection of discs $\{B_\alpha(w_j^{(1)}, t/8), B_\alpha(w_i^{(m)}, t/8)\}_{j=1,i=1}^{k_1,k_2}$ is a finite open cover for E_m. Continuing in this way we get a set of points $\{w_j\}_{j=1}^\infty$ in $\mathbb{D}$ such that the collection of the discs $U = \{B_\alpha(w_j, t/8)\}_{j=1}^\infty$ covers $\mathbb{D}$.

Pick a largest disc in U, say, $B_\alpha(w_1, t/8)$ and consider the disc $B_\alpha(w_1, t/3)$. Throw away all the discs from U which are contained in $B_\alpha(w_1, t/3)$ and name the remaining set $U^{(1)}$. Then $B_\alpha(w_1, t/3)$ and $U^{(1)}$ form again an open cover for $\mathbb{D}$. Now pick a largest disc from $U^{(1)}$ and let us rename it as $B_\alpha(w_2, t/8)$. Consider the disc $B_\alpha(w_2, t/3)$ and again throw away all the discs from $U^{(1)}$ which are contained in $B_\alpha(w_2, t/3)$. Rename the remaining set from $U^{(1)}$ as $U^{(2)}$. Then $B_\alpha(w_1, t/3)$, $B_\alpha(w_2, t/3)$ and $U^{(2)}$ form an open cover for $\mathbb{D}$. Continuing this process we get a new set of points $\{w_j\}_{j=1}^{\infty}$ in $\mathbb{D}$ and the related open cover (countable) $\{B_\alpha(w_j, t/3)\}_1^\infty$ for $\mathbb{D}$.

The cover constructed above satisfies the property that

$$B_\alpha\left(w_j, t/16\right) \cap B_\alpha\left(w_k, t/16\right) = \emptyset\,, \quad \forall \;\, j \neq k.$$

Indeed, we suppose contrary, *i.e.* $B_\alpha\left(w_j, t/16\right) \cap B_\alpha\left(w_k, t/16\right) \neq \emptyset$, for some $j \neq k$. WLOG we assume $|w_k| \leq |w_j|$. We claim

$$B_\alpha\left(w_j, t/8\right) \subset B_\alpha\left(w_k, t/3\right).$$

Indeed let $z \in B_\alpha\left(w_j, t/8\right)$. Then

$$\begin{aligned} |z - w_k| &\leq |z - w_j| + |w_j - w_k| \\ &< \frac{t(1-|z_j|)^\alpha}{8} + \left(\frac{t(1-|z_j|)^\alpha}{16} + \frac{t(1-|z_k|)^\alpha}{16}\right) \\ &< \frac{t(1-|z_k|)^\alpha}{3}\,. \end{aligned}$$

This implies

$$B_\alpha(w_j, t/8) \subset B_\alpha(w_k, t/3).$$

But this is not possible as $B_\alpha\left(w_j, t/8\right)$ is one of the discs which was thrown away in the construction of the open cover $\{B_\alpha(w_j, t/3)\}$. Henece our supposition is wrong.

The set of points $\{w_j\}_1^\infty$ constructed above is a (t, α)-lattice. In the rest of this paper $\{z_j\}$ is always a (t, α)-lattice. Correspondingly, we define $\rho_j = t(1 - |z_j|)^\alpha$ for all j and for a fixed $z \in \mathbb{D}$ define

$$J_z = \{j : z \in B_\alpha\left(z_j, t/2\right)\}\,.$$

Lemma 2.1. *Let $\{z_j\}$ be a (t, α)-lattice. The following two properties hold.*
P1- $\sum_j \chi_{B_\alpha\left(z_j, t/2\right)}(z) \leq M(t)$ *or equivalently* $|J_z| \leq M(t)$ *for all* $z \in \mathbb{D}$.
P2- *If* $z \in B_\alpha\left(z_j, t/2\right) \cap B_\alpha\left(z_k, t/2\right)$*, then*

$$B_\alpha\left(z, t/2^{\alpha+1}\right) \subset B_\alpha(z_j, t) \cap B_\alpha(z_k, t).$$

Here $M(t)$ is a positive integer and depends only on t and α.

Proof: The proof of **P2** is left to the reader. We only give the proof of **P1**.

First note that for fixed $z \in \mathbb{D}$, J_z is a finite set. Let

$$|z_i| = \min_{j \in J_z}(|z_j|).$$

Then $\rho_j \leq \rho_i$ for $j \in J_z$. Since

$$|z_j - z_i| \leq |z_j - z| + |z - z_i| < \tfrac{\rho_j}{2} + \tfrac{\rho_i}{2} \leq \rho_i \ , \quad \text{for } j \in J_z,$$

we have $z_j \in B_\alpha(z_i, t)$. Therefore

$$B_\alpha\left(z_j, t/16\right) \subseteq B_\alpha\left(z_i, (1 + 1/16)\, t\right) \ , \quad \text{for } j \in J_z.$$

Note that the discs $\{B_\alpha(z_j, t/16)\}_{j \in J_z}$ are disjoint. Hence the sum of the areas of $B_\alpha\left(z_j, t/16\right)$ for $j \in J_z$ is less than the area of $B_\alpha\left(z_i, (1 + 1/16)\, t\right)$, or equivalently

$$\sum_{j \in J_z} \left(\frac{\rho_j}{16}\right)^2 < \rho_i^2 \left(1 + \frac{1}{16}\right)^2 .$$

On the other hand, $|z_j - z_i| < \rho_i$ implies $|z_j| < \rho_i + |z_i|$ and hence

$$\rho_j = t(1 - |z_j|)^\alpha \quad > \quad t(1 - \rho_i - |z_i|)^\alpha > (1 - t)^\alpha \rho_i.$$

Therefore

$$\rho_i^2 \left(\frac{17}{16}\right)^2 > \sum_{j \in J_z} \left(\frac{\rho_j}{16}\right)^2 > \rho_i^2 \sum_{j \in J_z} \left(\frac{(1 - t)^\alpha}{16}\right)^2 .$$

Which implies

$$|J_z| < \frac{(17)^2}{(1 - t)^{2\alpha}}.$$

This finishes the proof of **P1**. $\square$

We will also need the following lemma which can be proved easily (see also [6]).

Lemma 2.2. *For a fixed $t \in (0, 1)$*

$$(1 - t)^\alpha \leq \frac{(1 - |w|)^\alpha}{(1 - |z|)^\alpha} \leq (1 + t)^\alpha \ , \quad \forall \ w \in B_\alpha(z, t), \ z \in \mathbb{D}.$$

3. Decomposition of BMO_α^p functions.

For $F \in L^1(\mathbb{D})$, let

$$||F||_{BMO_\alpha^p} = \sup_{z \in \mathbb{D}} \{MO_\alpha^p(F)(z)\}^{\frac{1}{p}},$$

and

$$||F||_{C^p} = \sup_{z \in \mathbb{D}} \left(\frac{1}{|B_\alpha(z, t)|} \int_{B_\alpha(z,t)} |F(w)|^p dA(w)\right)^{\frac{1}{p}} .$$

For F differentiable at z, we define the weighted gradient of F at z by

$$S(F)(z) = (1 - |z|)^\alpha \ |\nabla F(z)|.$$

For F in $C^1(\mathbb{D})$, denote

$$||F||_S = \sup_{z \in \mathbb{D}} \{(1 - |z|)^\alpha \ |\nabla F(z)|\}.$$

The following theorem gives us the decomposition of the BMO_α^p functions as described in the abstratct which has the root in Luecking's Paper [3] (see also [5]).

Theorem 3.1. *Suppose* $1 \leq p < \infty$ *and let* $F \in L^1(\mathbb{D})$. *Then* F *is in* BMO_α^p (VMO_α^p) *if and only if* $F = F_1 + F_2$ *such that*

(i)

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_1(w)|^p dA(w)$$

is bounded for all z *in* $\mathbb{D}$ (*tends to zero as* $|z|$ *tends to* 1 *for all* z *in* $\mathbb{D}$) ;

(ii) F_2 *is in* $L^{1,\infty}((1-|z|)^\alpha dA(z), \mathbb{D})$, *that is* F_2 *is in* $C^1(\mathbb{D})$ *and*

$$(1-|z|)^\alpha \ |\nabla F_2(z)|$$

is bounded for all z *in* $\mathbb{D}$ (*tends to zero as* $|z|$ *tends to* 1 *for all* z *in* $\mathbb{D}$) .

Furthermore

$$||F_1||_{C^p} \leq C||F||_{BMO_\alpha^p} \quad \text{and} \quad ||F_2||_S \leq C||F||_{BMO_\alpha^p} .$$

Proof: We prove "only if " part first. Select any fixed C_o^∞ function ψ with support in a disc centered at $z = 0$ of radius $\frac{1}{2}$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in a disc with center $z = 0$ and radius $\frac{1}{3}$. Let $H = \max_{z\in\mathbb{D}} |\nabla\psi(z)|$. Let $\{z_j\}$ be a (t, α)-lattice in $\mathbb{D}$ and set $\psi_j = \psi \circ \phi_{z_j}$ where $\phi_{z_j}(w) = \frac{w-z_j}{\rho_j}$. The function $\sum_k \psi_k$ is bounded away (for each $z \in \mathbb{D}$) from zero because $\{B_\alpha\left(z_j, \frac{t}{3}\right)\}$ covers $\mathbb{D}$. Finally, let

$$\gamma_j = \frac{\psi_j}{\sum_k \psi_k} \quad \forall\ j. \tag{3.1}$$

Clearly γ_j is in C^∞ and $\{\gamma_j\}$ is decomposition of unity. We claim that there exists $L > 0$ such that

$$S\left(\gamma_j\right)(z) = (1-|z|)^\alpha\, |\nabla\gamma_j(z)| \leq L\ , \quad \forall\ \ z \in\ \mathbb{D}. \tag{3.2}$$

Indeed, for fixed $z \in \mathbb{D}$ we have

$$\begin{aligned} \nabla(\gamma_j)(z) &= \nabla\left(\frac{\psi_j}{\sum_{k\in J_z}\psi_k}\right)(z) \\ &= \frac{\nabla\psi_j(z)\left(\sum_{k\in J_z}\psi_k(z)\right) - \psi_j(z)\left(\sum_{k\in J_z}\nabla\psi_k(z)\right)}{\left(\sum_{k\in J_z}\psi_k(z)\right)^2}. \end{aligned}$$

Since

$$1 \leq \left|\sum_{k\in J_z}\psi_k(z)\right| \leq |J_z|$$

and $|\nabla\psi_j(z)| \leq H$, we have

$$|\nabla\psi_j(z)| = \left|\nabla\psi\left(\frac{z-z_j}{\rho_j}\right)\right| \frac{1}{\rho_j} \leq \frac{H}{\rho_j} .$$

Therefore

$$|\nabla(\gamma_j)(z)| \leq \left(\frac{H}{\rho_j}\right)|J_z| + (1)\left(\sum_{k\in J_z}\frac{H}{\rho_k}\right) .$$

Since J_z is finite set, we can can select $i \in J_z$ such that $|z_k| \leq |z_i|$ for all $k \in J_z$ and hence $\rho_i \leq \rho_k$ for all $k \in J_z$. Then

$$|\nabla(\gamma_j)(z)| \leq \left(\frac{H}{\rho_i}\right)|J_z| + \frac{H}{\rho_i}|J_z| = (|J_z|+1)\,\frac{H}{\rho_i}. \tag{3.3}$$

On the other hand, for any $j \in J_z$, we have

$$|z-z_j| < \frac{\rho_j}{2}.$$

Hence

$$1-|z| \;\; \leq \;\; 1-|z_j| + |z-z_j| < \frac{3}{2}\left(1-|z_j|\right).$$

Therefore

$$\frac{(1-|z|)^\alpha}{\rho_j} < \frac{1}{t}\left(\frac{3}{2}\right)^\alpha , \quad \forall\, j \in J_z.$$

Using above inequality in (3.3) together with the fact that $|J_z| \leq M(t)$, we get

$$S(\gamma_j)(z) = (1-|z|)^\alpha\,|\nabla(\gamma_j)(z)| \leq H\,(M(t)+1)\;\frac{1}{t}\left(\frac{3}{2}\right)^\alpha .$$

This finishes the proof of our calim in (3.2).

For simplification, denote $B_j = B_\alpha\left(z_j, t/2\right)$ and $c_j = \frac{1}{|B_j|}\int_{B_j} F dA$. We have

$$\frac{1}{|B_j|}\int_{B_j}|F-c_j|^p dA = MO_\alpha^p\left(F\right)\left(z_j, t/2\right).$$

Let

$$F_2(z) = \sum_j c_j\gamma_j(z) \quad \text{ and } \quad F_1 = F - F_2 .$$

We only need to show that F_1 and F_2 satisfy the conditions stated in the theorem.

Indeed, for any disc $B_\alpha(z,t)$ in $\mathbb{D}$, we have, using the fact that $\sum_k \gamma_k \equiv 1$

$$\begin{aligned}
&\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_1|^p dA\right)^{\frac{1}{p}} \\
&\quad = \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|\sum_{k\in J_w}(F-c_k)\gamma_k\right|^p dA(w)\right)^{\frac{1}{p}} \\
&\quad \le \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left(\left(\sum_{k\in J_w}|F-c_k|^p\right)^{\frac{1}{p}}\left(\sum_{k\in J_w}|\gamma_k|^q\right)^{\frac{1}{q}}\right)^p dA(w)\right)^{\frac{1}{p}}.
\end{aligned}$$

Let $\widetilde{J}_z = \cup_{w\in B_\alpha(z,t)} J_w$. It is easy to see that $\widetilde{J}_z$ is a finite set. And by the similar argument as in Lemma 2.1, we can prove that $|\widetilde{J}_z| \le \widetilde{M}(t)$ with $\widetilde{M}(t)$ independant of z. Also $|\gamma_j| \le 1$ and J_w is finite ($|J_w| \le M(t)$). Therefore from above inequality we get

$$\begin{aligned}
\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_1|^p dA\right)^{\frac{1}{p}} &\le \frac{(M(t))^{\frac{1}{q}}}{|B_\alpha(z,t)|^{\frac{1}{p}}}\left(\int_{B_\alpha(z,t)\cap B_j}\sum_{k\in\tilde{J}_z}|F-c_k|^p dA\right)^{\frac{1}{p}} \\
&\le (M(t))^{\frac{1}{q}}\left(\sum_{k\in\tilde{J}_z}\frac{|B_k|}{|B_\alpha(z,t)|}MO_\alpha^p(F)\,(z_k,t/2)\right)^{\frac{1}{p}} \\
&\le (M(t))^{\frac{1}{q}}\left(\widetilde{M}(t)\right)^{\frac{1}{p}}\max_{k\in\tilde{J}_z}\frac{|B_k|}{|B_\alpha(z,t)|}||F||_{BMO_\alpha^p}.
\end{aligned}$$

It is easy to prove that $|B_k| \asymp |B_\alpha(z,t)|$ for $k \in \widetilde{J}_z$. Hence F_1 satisfies the Carleson condition stated in the theorem.

To show that F_2 satifies the Sobolev condition, we need the following pointwise estimate of $\nabla F_2(z)$. Fix a point z in $\mathbb{D}$, we have

$$F_2(z) = \sum_j c_j\gamma_j(z) \quad .$$

For convenience suppose that $1 \in J_z$. Now we can write

$$F_2(z) = c_1 + \sum_j (c_j - c_1)\gamma_j(z) \quad .$$

Therefore

$$\nabla F_2(z) = \sum_j (c_j - c_1)\nabla\gamma_j(z) = \sum_{j\in J_z}(c_j - c_1)\nabla\gamma_j(z) \quad .$$

This together with (3.2) yeilds

$$S\left(F_2\right)(z) \leq \sum_{j\in J_z} |c_j - c_1| S(\gamma_j)(z) \leq L \sum_{j\in J_z} |c_j - c_1|. \tag{3.4}$$

Let $j \in J_z$. Since $z \in B_j \cap B_1$, we have (by **P2**)

$$B_\alpha\left(z, t/2^{\alpha+1}\right) \subseteq B_\alpha(z_j, t) \cap B(z_1, t).$$

Let $a = t/2^{\alpha+1}$. Then

$$\begin{aligned}
|c_j - c_1| &= \left(\frac{1}{|B_\alpha(z,a)|}\int_{B_\alpha(z,a)} |c_j - c_1|^p dA\right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{|B_\alpha(z,a)|}\int_{B_\alpha(z,a)} \left(|c_j - F| + |F - c_1|\right)^p dA\right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{|B_\alpha(z,a)|}\int_{B_\alpha(z_j,t)} |c_j - F|^p dA\right)^{\frac{1}{p}} + \left(\frac{1}{|B_\alpha(z,a)|}\int_{B_\alpha(z_1,t)} |F - c_1|^p dA\right)^{\frac{1}{p}} \\
&\leq \left|\frac{B_\alpha(z_j,t)}{B_\alpha(z,a)}\right|^{\frac{1}{p}} MO_\alpha^p(F)(z_j) + \left|\frac{B_\alpha(z_1,t)}{B_\alpha(z,a)}\right|^{\frac{1}{p}} MO_\alpha^p\left(F\right)(z_1).
\end{aligned}$$

Therefore (3.4) yeilds

$$\begin{aligned}
S\left(F_2\right)(z) &\leq L\sum_{j\in J_z}\left(\left|\frac{B_\alpha(z_j,t)}{B_\alpha(z,a)}\right|^{\frac{1}{p}} MO_\alpha^p(F)(z_j) + \left|\frac{B_\alpha(z_1,t)}{B_\alpha(z,a)}\right|^{\frac{1}{p}} MO_\alpha^p(F)(z_1)\right) \\
&\leq C\, ||F||^p_{BMO_\alpha^p}.
\end{aligned}$$

We now prove the "if" part of the theorem. It is easy to see that F_1 is in BMO_α^p. For F_2, we have

$$\begin{aligned}
&\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|F_2 - \widehat{F}_2(z)\right|^p dA(w) \\
&\quad= \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|F_2 - \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} F_2(w_1)dA(w_1)\right|^p dA(w) \\
&\quad= \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left(F_2(w) - F_2(w_1)\right) dA(w_1)\right|^p dA(w)\ .
\end{aligned}$$

By Poincaré inequality we have

$$\begin{aligned}
\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|F_2 - \widehat{F}_2(z)\right|^p dA(w) &\leq \frac{C\,[t(1-|z|)^\alpha]^p}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\nabla F_2|^p dA(w) \\
&\leq \frac{C\,[t(1-|z|)^\alpha]^p}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \frac{[S(F_2)(w)]^p}{(1-|w|)^{\alpha p}} dA(w).
\end{aligned}$$

By Lemma 2.2, we have

$$\frac{1}{(1-|w|)^{\alpha p}} \le \frac{1}{(1-t)^{\alpha p}\,(1-|z|)^{\alpha p}}.$$

Therefore from above inequality, we get

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)}\left|F_2-\widehat{F}_2(z)\right|^p dA(w) \le C\frac{t^p}{(1-t)^{\alpha p}}\sup_{w\in B_\alpha(z,t)} S(F_2)(w) \le C\|F_2\|_S.$$

This finishes the proof and hence the theorem. □

4. More Properties of BMO^p_α Spaces and α-Bloch Spaces.

In the rest of this paper we denote $d\mu = c_\alpha\,(1-|z|)^{\alpha-1}\,dA(z)$ where c_α is such that $\int d\mu = 1$. The following two lemmas are direct application of Theorem 3.1. Similar results can be found in [6].

Lemma 4.1. *Suppose F is locally integrable on $\mathbb{D}$. If $MO^p_\alpha(F)\in L^\infty$, then $F\in L^1(\mu)$.*

Proof: By Theorem 3.1, we can write $F=F_1+F_2$ with both functions $M^p_\alpha(F_1)$ and $S(F_2)$ bounded on $\mathbb{D}$. Now we only need to show that both F_1 and F_2 are in $L^1(\mu)$. We claim that

$$\left(\frac{1-t}{1+t}\right)^{2\,\alpha} \le \int\frac{\chi_{B_\alpha(z,t)}(w)}{|B_\alpha(z,t)|}dA(z) \le \left(\frac{1+t}{1-t}\right)^{2\,\alpha}.$$

We will make use of Lemma 2.2 to prove above inequality. Indeed, consider

$$\begin{aligned}
\int\frac{\chi_{B_\alpha(z,t)}(w)}{|B_\alpha(z,t)|}dA(z) &= \int\frac{\chi_{B_\alpha(z,t)}(w)}{t^2(1-|z|)^{2\,\alpha}}dA(z)\\
&\le \frac{(1+t)^{2\,\alpha}}{t^2(1-|w|)^{2\,\alpha}}\int_{|z-w|<t(1-|z|)^\alpha} dA(z)\\
&\le \frac{(1+t)^{2\,\alpha}}{t^2(1-|w|)^{2\,\alpha}}\int_{|z-w|<\frac{t}{(1-t)^\alpha}(1-|w|)^\alpha} dA(z)\\
&= \frac{(1+t)^{2\,\alpha}}{t^2(1-|w|)^{2\,\alpha}}\left(\frac{t}{(1-t)^\alpha}(1-|w|)^\alpha\right)^2\\
&= \left(\frac{1+t}{1-t}\right)^{2\,\alpha}.
\end{aligned}$$

This proves the upper estimate. The lower estimate can be proved similarly.

We are now ready to eastimate the $L^1(\mu)$ norms of F_1 and F_2. For F_1 we have

$$\begin{aligned}
\int |F_1| d\mu &\leq \left(\frac{1+t}{1-t}\right)^{2\,\alpha} \int \left(\int \frac{\chi_{B_\alpha(z,t)}(w)}{|B_\alpha(z,t)|} dA(z)\right) |F_1(w)| d\mu(w) \\
&= c_\alpha \left(\frac{1+t}{1-t}\right)^{2\,\alpha} \int \left(\int \frac{\chi_{B_\alpha(z,t)}(w)|F_1(w)|(1-|w|)^{\alpha-1}}{|B_\alpha(z,t)|} dA(w)\right) dA(z) \\
&\leq c_\alpha (1+t)^{\alpha-1} \left(\frac{1+t}{1-t}\right)^{2\,\alpha} \int M_\alpha^1(F_1)(z)(1-|z|)^{\alpha-1} dA(z) \\
&\leq \frac{(1+t)^{3\,\alpha-1}}{(1-t)^{2\,\alpha}} \|M_\alpha^1(F_1)(z)\|_\infty \int d\mu \\
&\leq \frac{(1+t)^{3\,\alpha-1}}{(1-t)^{2\,\alpha}} ||M_\alpha^p(F_1)(z)||_\infty .
\end{aligned}$$

For F_2, first we note that for $w \in \mathbb{D}$

$$\begin{aligned}
|F_2(w) - F_2(0)| &= \left| \int_0^1 \frac{d}{ds}(F_2(sw)) w ds \right| \\
&\leq |w| \int_0^1 |\nabla F_2(sw)| \, ds \\
&\leq ||S(F_2)||_\infty |w| \int_0^1 \frac{1}{(1-s|w|)^\alpha} \, ds \\
&\leq C\, ||S(F_2)||_\infty \frac{1}{(1-|w|)^{\alpha-1}} \, .
\end{aligned}$$

Therefore

$$\begin{aligned}
\int |F_2| \, d\mu &\leq \int |F_2 - F_2(0)| \, d\mu + |F_2(0)| \\
&\leq C\, ||S(F_2)||_\infty \int \frac{1}{(1-|w|)^{\alpha-1}} \, d\mu + |F_2(0)| \\
&\leq C\, ||S(F_2)||_\infty + |F_2(0)| \, . \qquad \square
\end{aligned}$$

Lemma 4.1 allows us to modify the norm for BMO_α^p space in the following way so that BMO_α^p is not a module.

$$\|F\|_{BMO_\alpha^p} = \|F\|_{L^1(\mu)} + \|MO_\alpha^p(F)\|_\infty \, .$$

We will use this norm for BMO_α^p in the rest of this paper. The following lemma estimates the size of $\widehat{F}(z)$ for $F \in BMO_\alpha^p$.

Lemma 4.2. *Suppose $p \geq 1$ and $f \in BMO_\alpha^p$. Then*

$$|\widehat{F}(z)| \leq \frac{C\, \|F\|_{BMO_\alpha^p}}{(1-|z|)^{\alpha-1}} \, , \quad \forall \;\; z \in \mathbb{D}$$

where C is a positive constant independent of p.

Proof: By Theorem 3.1, there is a decomposition $F = F_1 + F_2$ such that

$$\|F\|_{L^1(\mu)} + \|M^p_\alpha(F_1)\|_\infty + \|S(F_2)\|_\infty \le C\,||F||_{BMO^p_\alpha}$$

where $C \ge 1$ is a constant.

In order to estimate $|\widehat{F}(z)|$, consider the following decomposition

$$\begin{aligned}
\widehat{F}(z) &= \int F d\mu(w) - \int F d\mu(w) - \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_2(0)|\, dA(w) \\
&\quad + \int |F_2(0)|\, d\mu(w) + \widehat{F_1}(z) + \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} F_2(w)\, dA(w) \\
&= \int F\, d\mu(w) + \widehat{F_1}(z) - \int F_1\, d\mu(w) - \int F_2\, d\mu(w) \\
&\quad + \int F_2(0)\, d\mu(w) + \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} F_2(w)\, dA(w) \\
&\quad - \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} F_2(0)\, dA(w)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\widehat{F}(z)| &\le \|F\|_{L^1(\mu)} + |\widehat{F_1}(z)| + \|F_1\|_{L^1(\mu)} + \int |F_2 - F_2(0)|\, d\mu(w) \\
&\quad + \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_2(w) - F_2(0)|\, dA(w) \\
&\le \|F\|_{L^1(\mu)} + |\widehat{F_1}(z)| + \|F_1\|_{L^1(\mu)} \\
&\quad + \|F_2 - F_2(0)\|_{L^1(\mu)} + M^1_\alpha\left(F_2 - F_2(0)\right)(z)\,. \qquad (4.5)
\end{aligned}$$

By the proof of Lemma 4.1, we have

$$\|F\|_{L^1(\mu)} \le C\,\|M^p_\alpha(F_1)\|_\infty \quad , \quad \|F_2 - F_2(0)\|_{L^1(\mu)} \le C\|S(F_2)\|_\infty \qquad (4.6)$$

and

$$|F_2 - F_2(0)| \le \frac{C\,\|S(F_2)\|_\infty}{(1-|w|)^{\alpha-1}}.$$

Therefore

$$\begin{aligned}
M^1_\alpha\left(F_2 - F_2(0)\right)(z) &= \frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |F_2(w) - F_2(0)|\, dA(w) \\
&\le \frac{C\,\|S(F_2)\|_\infty}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \frac{1}{(1-|w|)^{\alpha-1}}\, dA(w) \\
&\le \frac{C\,\|S(F_2)\|_\infty}{(1-|z|)^{\alpha-1}}\,. \qquad (4.7)
\end{aligned}$$

Therefore by (4.5), (4.6) and (4.7), we have

$$\begin{aligned}|\widehat{F}(z)| &\leq \|F\|_{L^1(\mu)} + M^p_\alpha(F_1)(z) + C\,\|M^p_\alpha(F_1)\|_\infty + C\|S(F_2)\|_\infty + \frac{C\,\|S(F_2)\|_\infty}{(1-|z|)^{\alpha-1}} \\ &\leq \frac{C\,\|F\|_{BMO^p_\alpha}}{(1-|z|)^{\alpha-1}}\,.\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 4.3. *Suppose $p \geq 1$ and K is a compact set in $\mathbb{D}$. If $F \in L^p$ and $F = 0$ a.e. on $\mathbb{D}\backslash K$, then $F \in VMO^p_\alpha$ and*

$$||F||_{BMO^p_\alpha} \leq \frac{2 + c_\alpha}{\inf\ |B_\alpha(z,t)|^{\frac{1}{p}}}\|F\|_{L^p}$$

where the infimum is taken over all $w \in \mathbb{D}$ such that $B_\alpha(w) \cap K \neq \emptyset$.

Proof: It is easy to prove that for $z \in \mathbb{D}$, we have

$$\begin{aligned}MO^p_\alpha(F)(z) &\leq 2\,M^p_\alpha(F)(z) \\ &= 2\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)\cap K}|F|^p dA\right)^{\frac{1}{p}} \\ &\leq 2\left(\frac{1}{\inf|B_\alpha(z,t)|}\int_{B_\alpha(z,t)\cap K}|F|^p dA\right)^{\frac{1}{p}} \\ &\leq \frac{2}{\inf|B_\alpha(z,t)|^{\frac{1}{p}}}\|F\|_{L^p}.\end{aligned}$$

Also, note that the second estimate above suggests that $MO^p_\alpha(F)(z) = 0$ as z is sufficiently get close to the boundary of $\mathbb{D}$, since then $B_\alpha(z,t) \cap K = \phi$. (Because then $B_\alpha(z,t)$ will be very small). Therefore $f \in VMO^p_\alpha$. We further note that $|B_\alpha(z,t)| \leq 1$. Hence

$$\|F\|_{L^1(\mu)} \leq c_\alpha\|F\|_{L^1} \leq c_\alpha\|F\|_{L^p} \leq \frac{c_\alpha}{\inf|B_\alpha(z,t)|^{\frac{1}{p}}}\|F\|_{L^p}$$

and the desired result follows. □

Remark: Note that if z_o is a point in K as in Lemma (4.3) such that $|z_o| = \sup_{w\in K}|w|$, then

$$\inf_{w\in\mathbb{D},\ B_\alpha(w)\cap K\neq\phi}|B_\alpha(w)| = \inf_{w\in\mathbb{D},\ z_o\in B_\alpha(w)}|B_\alpha(w)| \asymp |B_\alpha(z_o,t)|\,.$$

The following lemma shows that $MO^p_\alpha(F)(z)$ and $S(F)(z)$ are compareable if F is harmonic on $\mathbb{D}$.

Lemma 4.4. *Suppose $z \in \mathbb{D}$, $p \geq 1$ and $F \in L^1_{loc}(\mathbb{D})$.*

(i) *If F is differentiable on $B_\alpha(z,t)$, then*

$$MO^p_\alpha(F)(z) \le \frac{2t}{(1-t)^\alpha} \sup_{w\in B_\alpha(z,t)} S(F)(w).$$

(ii) *If F is harmonic on $B_\alpha(z,t)$, then*

$$MO^p_\alpha(F)(z) \ge \frac{t}{4\sqrt{2}}\, S(F)(z).$$

Proof:

For (i), we consider $u \in B_\alpha(z,t)$ and write $u = \xi + z$ with $|\xi| < t(1-|z|)^\alpha$. Now

$$\begin{aligned} |F(u) - F(z)| &= \left| \int_0^1 \frac{d}{ds} F(s\xi + z)\xi \, ds \right| \\ &\le t(1-|z|)^\alpha \int_0^1 |\nabla F(s\xi+z)| \, ds \\ &\le t(1-|z|)^\alpha \sup_{w\in B_\alpha(z,t)} |\nabla F(w)| \\ &\le \frac{t}{(1-t)^\alpha} \sup_{w\in B_\alpha(z,t)} (1-|w|)^\alpha |\nabla F(w)| \, . \end{aligned}$$

Therefore

$$MO^p_\alpha(F)(z) \le 2\, M^p_\alpha(F - F(z))(z) \le \frac{2\,t}{(1-t)^\alpha} \sup_{w\in B_\alpha(z,t)} S(F)(w).$$

For (ii) we use the harmonicity of F in that the mean of F over the disc equals the value of F at the center of the disc, *i.e.*

$$F(z) = \frac{1}{|B_\alpha(z,t)|} \int_{B_\alpha(z,t)} F(w)\,dA(w) = \widehat{F}(z).$$

Also, let $f = F + i\widetilde{F}$ where $\widetilde{F}$ is a conjugate harmonic of F. Recall that the reproducing formula for analytic function $g \in L^1$ is

$$g(z) = \int g(w) \frac{1}{(1-\overline{w}\, z)^2} dA(w) \, .$$

We have

$$g(z) - g(0) = \int \frac{g(w) - g(0)}{(1-\overline{w}\, z)^2} dA(w).$$

Therefore

$$(\nabla g)\,(0) = 2 \int (g(w) - g(0))\, \overline{w} \, < 1, i > \, dA(w) \, .$$

Note that

$$\int \left(\overline{g(w)} - \overline{g(0)} \right) \overline{w} \, < 1, i > \, dA(w) \, = 0.$$

Therefore

$$(\nabla g)(0) = \int (\mathcal{R}e(g(w)) - \mathcal{R}e(g(0)))\, \overline{w} \ < 1, i > \ dA(w).$$

Now translating this formula onto the disc $\{w \ : \ |w| < t(1-|z|)^\alpha\}$, we have

$$\begin{aligned} & t(1-|z|)^\alpha \nabla f(z) \\ = \ & \frac{4}{(t(1-|z|)^\alpha)^3} \int_{|w|<t(1-|z|)^\alpha} (\mathcal{R}e(f(w+z)) - \mathcal{R}e(f(z))) \ \overline{w} \ < 1, i > \ dA(w) \\ = \ & \frac{4}{(t(1-|z|)^\alpha)^3} \int_{|w|<t(1-|z|)^\alpha} \Big((F(w+z)) - (\widehat{F}(z)) \Big) \ \overline{w} \ < 1, i > \ dA(w). \end{aligned}$$

Therefore

$$\begin{aligned} t(1-|z|)^\alpha \, |\nabla F(z)| \ & \leq \ t(1-|z|)^\alpha \, |\nabla f(z)| \\ & \leq \ \frac{4\sqrt{2}}{(t(1-|z|)^\alpha)^3} \int_{|w|<t(1-|z|)^\alpha} \left| F(w+z) - \widehat{F}(z) \right| \ |\overline{w}| \ dA(w) \\ & \leq \ \frac{4\sqrt{2}}{(t(1-|z|)^\alpha)^2} \int_{|w|<t(1-|z|)^\alpha} \left| F(w+z) - \widehat{F}(z) \right| \ dA(w) \\ & \leq \ 4\sqrt{2} \left(\frac{1}{|B_\alpha(z,t)|} \int_{B_\alpha(z,t)} \left| F(w+z) - \widehat{F}(z) \right|^p \ dA(w) \right)^{\frac{1}{p}} \\ & = \ 4\sqrt{2}\, MO_\alpha^p(F)(z). \end{aligned}$$

That is

$$\frac{t}{4\sqrt{2}} \, S(F)(z) \leq MO_\alpha^p(F)(z).$$

This finishes the proof of part (ii). □

Corollary 4.5. *The harmonic BMO_α^p (or VMO_α^p) space with $p \geq 1$ is (equivalent to) the harmonic α-Bloch space (or little α-Bloch space), which is the space of harmonic functions F on $\mathbb{D}$ with the norm*

$$\|F\|_{B_\alpha} = |F(0)| + \sup_{z \in \mathbb{D}} (1-|z|)^\alpha |\nabla F(z)|$$

(or $\|F\|_{B_\alpha} < \infty$, $(1-|z|)^\alpha \, |\nabla F(z)| \to 1$ as $|z| \to 1$).

Proof: Since

$$||F||_{BMO_\alpha^p} = \|F\|_{L^1(\mu)} + \|MO_\alpha^p(F)\|_\infty.$$

and by Mean-Value Theorem $|F(0)| \le \|F\|_{L^1(\mu)}$, we only need to estimate $\|F\|_{L^1(\mu)}$. By the proof of Lemma 4.1, we have

$$\begin{aligned} \|F\|_{L^1(\mu)} \le |F(0)| + \|F - F(0)\|_{L^1(\mu)} &\le |F(0)| + C\|S(F)\|_\infty \int \frac{1}{(1-|w|)^{\alpha-1}}\, d\mu \\ &= |F(0)| + C\|S(F)\|_\infty \,. \end{aligned}$$

Now using Lemma 4.4, we can conclude $\|F\|_{B_\alpha} \asymp ||F||_{BMO_\alpha^p}$. □

5. **Multipliers of BMO_α^p Space.**

As before, the norm in BMO_α^p (also VMO_α^p) is given by

$$||F||_{BMO_\alpha^p} = \|F\|_{L^1(\mu)} + \|MO_\alpha^p(F)\|_\infty.$$

Consider the multiplier problem:

$$\phi BMO_\alpha^p \subset BMO_\alpha^q.$$

It is standard that, by the closed graph theorem, ϕ is a multiplier from BMO_α^p to BMO_α^q if and only if there is a constant $C = C(p,q) > 0$ such that

$$\|\phi F\|_{BMO_\alpha^q} \le \ C \ \|F\|_{BMO_\alpha^p} \,, \quad \forall\, F \in BMO_\alpha^p.$$

We denote the smallest constant C here by $\|\phi\|_{p,q}$. A multiplier from VMO_α^p to VMO_α^q is defined in a similar way. For $\alpha = 1$, the multiplier problem has been studied in [6]. The results in this section are generalizations of those in [6]. The main results of this section are as follows:

Theorem 5.1. *Let $\phi \in L^1_{loc}(\mathbb{D})$.*

(i) *Suppose $1 \le p < q < \infty$. Then ϕ is a multiplier from BMO_α^p to BMO_α^q (or VMO_α^p to VMO_α^p) if and only if $\phi = 0$, a.e. on $\mathbb{D}$.*

(ii) *Suppose $1 \le p = q < \infty$. Then ϕ is a multiplier from BMO_α^p to BMO_α^p (or VMO_α^p to VMO_α^p) if and only if both functions $\phi(z)$ and $\frac{1}{(1-|z|)^{\alpha-1}}\, MO_\alpha^p(\phi)(z)$ are bounded on $\mathbb{D}$. Moreover, the operator norm of ϕ is equivalent to*

$$\|\phi\|_\infty + \left\| \frac{1}{(1-|z|)^{\alpha-1}}\, MO_\alpha^p(\phi)(z) \right\|_\infty .$$

(iii) *Suppose $1 \le q < p < \infty$. Then ϕ is a multiplier from BMO_α^p to BMO_α^q (or VMO_α^p to VMO_α^q) if and only if both functions $\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi|^{\frac{pq}{p-q}}\ dA\right)^{\frac{p-q}{pq}}$ and $\frac{1}{(1-|z|)^{\alpha-1}}\ MO_\alpha^q(\phi)(z)$ are bounded on $\mathbb{D}$. Moreover, the operator norm of ϕ is equivalent to*

$$\left\| \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi|^{\frac{pq}{p-q}}\ dA\right)^{\frac{p-q}{pq}} \right\|_\infty + \left\| \frac{1}{(1-|z|)^{\alpha-1}}\ MO_\alpha^q(\phi)(z) \right\|_\infty .$$

(iv) *Suppose $1 \le q \le q < \infty$, and ϕ is harmoic on $\mathbb{D}$. Then ϕ is a multiplier from BMO_α^p to BMO_α^q (or VMO_α^p to VMO_α^q) if and only if both functions $\phi(z)$ and $(1-|z|)|\nabla\phi(z)|$ are bounded on $\mathbb{D}$. Moreover, the operator norm of ϕ is equivalent to*

$$\|\phi\|_\infty + \sup_{z\in\mathbb{D}} \left((1-|z|)|\nabla\phi(z)|\right) .$$

Note: We remark that the results obtained for the case $\alpha = 1$ in [6] are different than the one in Theorem 5.1. The conditions in [6] for ϕ to be a multilpier involve a factor $\log\frac{e}{1-|z|}$. But the conditions in Theorem 5.1 do help one to understand the relation betwwen the case $\alpha > 1$ and $\alpha = 1$.

The following Corollary can be obtained by the last part of Theorem 5.1 which is well known.

Corollary 5.2. *Suppose ϕ is analytic on $\mathbb{D}$. Then ϕ is a multiplier of α-Bloch space if and only if $\phi \in H^\infty(\mathbb{D})$ and*

$$(1-|z|)|\phi'(z)|$$

is bounded for all z in $\mathbb{D}$. Moreover, the operator norm of ϕ is equivalent to

$$\|\phi\|_{H^\infty} + \|(1-|z|)|\phi'(z)|\|_\infty .$$

To be able to prove the theorems, we first investigate the "local" behaviour of the q-mean oscillation of ϕF at $z \in \mathbb{D}$.

Lemma 5.3. *Suppose $z \in \mathbb{D}$ and $1 \le p < q < \infty$. If there exists $N > 0$ such that $MO_\alpha^q(\phi F)(z) \ \le\ N\ ||F||_{BMO_\alpha^p}$ for all $F \in VMO_\alpha^p$, then $\phi = 0$ a.e. on $B_\alpha(z,t)$.*

Proof: Assume that ϕ is not zero a.e. on $B_\alpha(z,t)$. Then there is a positive number $\delta > 0$ such that the area of the set

$$C = \{w \in B_\alpha(z,t)\ :\ |\phi(w)|\ \ge \delta > 0\}$$

is not zero. Now we construct the following test functions in BMO_α^p to show that this is impossible.

Let $\{F_m\}$ be a sequence of functions on $\mathbb{D}$ so that $F_m \in L^p,\ F_m = 0$ on $\mathbb{D}\backslash F$, and $\|F_m\|_{L^p} \leq 1$, but $\|F_m\|_{L^q} \to \infty$ as $m \to \infty$. We calim that there exists a half-disc E_m of $B_\alpha(z,t)$ so that

$$\int_{E_m} \phi F_m dA(z) = \int_{B_\alpha(z,t)\backslash E_m} \phi F_m dA(z).$$

For, if E is the half-disc of $B_\alpha(z,t)$ then the function

$$I(E) = \int_{E_m} \phi F_m dA(z) - \int_{B_\alpha(z,t)\backslash E_m} \phi F_m dA(z)$$

is a continuous function of the position of E in $B_\alpha(z,t)$. Since $I(E) = -I(B_\alpha(z,t)\backslash E)$, by the Intermediate-Value Theorem of continuous functions, there exists a half-disc E_m such that $I(E_m) = 0$. Now we can define our test functions as follows:

$$G_m = F_m\left(\chi_{E_m} - \chi_{B_\alpha(z,t)\backslash E_m}\right).$$

Since $|G_m| = |F_m|$, we have $\|G_m\|_{L^p} = \|F_m\|_{L^p} \leq 1$ and $\|G_m\|_{L^q} = \|F_m\|_{L^q}$. Moreover, since $supp(G_m) = \ supp(F_m) \subset B_\alpha(z,t)$, by the Lemma 4.3 and its remark $G_m \in VMO^p_\alpha$ and

$$\|G_m\|_{BMO^p_\alpha} \leq \frac{3\|G_m\|_{L^p}}{\inf\{|B_\alpha(w,t)|^{\frac{1}{p}}\ :\ B_\alpha(w,t)\cap\overline{B_\alpha(z,t)} \neq \phi\}} \leq 3\left(\frac{1+t}{1-t}\right)^{\frac{2\alpha}{p}} \frac{1}{|B_\alpha(z,t)|^{\frac{1}{p}}}.$$

The key property of our test function is that $\widehat{\phi G_m}(z) = 0$, which is easy to check. We therefore have

$$\begin{aligned}
\|G_m\|_{L^q} &= |B_\alpha(z,t)|^{\frac{1}{q}}\left(\frac{1}{|B_\alpha(z,t)|}\int |G_m|^q dA(w)\right)^{\frac{1}{q}} \\
&\leq \frac{|B_\alpha(z,t)|^{\frac{1}{q}}}{\delta}\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi|^q|G_m|^q dA(w)\right)^{\frac{1}{q}} \\
&= \frac{|B_\alpha(z,t)|^{\frac{1}{q}}}{\delta}\ M^q_\alpha(\phi G_m)(z) \\
&= \frac{|B_\alpha(z,t)|^{\frac{1}{q}}}{\delta}\ MO^q_\alpha(\phi G_m)(z) \\
&\leq \frac{|B_\alpha(z,t)|^{\frac{1}{q}}}{\delta}\ N\ \|G_m\|_{BMO^p_\alpha} \\
&\leq \frac{3N}{\delta}\left(\frac{1+t}{1-t}\right)^{\frac{2\alpha}{p}} |B_\alpha(z,t)|^{\frac{1}{q}-\frac{1}{p}}.
\end{aligned}$$

This contracdicts the fact that the sequence $\{G_m\}$ is unbounded. $\square$

Corollary 5.4. *Suppose* $1 \leq p < q < \infty$*. If* $\phi VMO^p_\alpha \subset BMO^q_\alpha$ *, then* $\phi = 0$ *a.e. on* $\mathbb{D}$*.*

Lemma 5.5. *Suppose* $z \in \mathbb{D}$ *and* $1 \leq p < q < \infty$*. If there exists* $N > 0$ *such that* $MO^q_\alpha(\phi F)(z) \leq N \, ||F||_{BMO^p_\alpha}$ *for all* $F \in VMO^p_\alpha$*, then*

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi|^{\frac{pq}{p-q}} dA \leq (CN)^{\frac{pq}{p-q}}$$

where C *ia positive constant independant of* p *and* q*.*

Proof: For $T > 0$, let $\phi_T = \max(-T, \min(T, \phi))$, the truncation of ϕ. Consider the test function

$$f_T = |\phi_T|^{\frac{q}{p-q}} \; Sign(\phi) \; \left(\chi_{E_T} - \chi_{B_\alpha(z,t)\setminus E_T}\right)$$

where $E_T\,(\subset B_\alpha(z,t))$ is a half-disc, choosen as in the proof of Lemma 5.3, so that $\widehat{\phi f_T} = 0$. Then

$$\|f_T\|_p^p = \int_{\mathbb{D}} |f_T|^p dA = \int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}} dA \, .$$

Since $supp f_T \subset B_\alpha(z,t)$, we have, by Lemma 4.3 and its remark, $f_T \in VMO^p_\alpha$ and

$$\begin{aligned}
\|f_T\|_{BMO^p_\alpha} &\leq \frac{(2 + c_\alpha)\;\|f_T\|_p}{\inf |B_\alpha(w)|^{\frac{1}{p}}} \\
&= (2 + c_\alpha)\;\left(\frac{1+t}{1-t}\right)^{2\,\alpha} \frac{\|f_T\|_p}{|B_\alpha(z,t)|^{\frac{1}{p}}} \\
&= (2 + c_\alpha)\;\left(\frac{1+t}{1-t}\right)^{2\,\alpha} \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\; dA\right)^{\frac{1}{p}} . \qquad (5.8)
\end{aligned}$$

On the other hand since $|\phi_T|^{\frac{p}{p-q}} \leq |\phi|\;|f_T|$, we have

$$\begin{aligned}
\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\; dA\right)^{\frac{1}{q}} &\leq \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi|^q |f_T|^q\; dA\right)^{\frac{1}{q}} \\
&= MO^q_\alpha(\phi f_T)(z) \\
&\leq N\;\|f_T\|_{BMO^p_\alpha} .
\end{aligned}$$

Therefore by (5.8)

$$\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\; dA\right)^{\frac{1}{q}} \leq 3\;N\;\left(\frac{1+t}{1-t}\right)^{2\,\alpha} \left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\; dA\right)^{\frac{1}{p}} .$$

Which implies

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\; dA \leq \left(3\;N\;\left(\frac{1+t}{1-t}\right)^{2\,\alpha}\right)^{\frac{pq}{p-q}} .$$

Letting $T \to \infty$, we get the desired estimate. $\square$

Corollary 5.6. *Suppose* $1 \le q < p < \infty$*. If* $\phi VMO_\alpha^p \subset BMO_\alpha^q$*, then*

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} |\phi_T|^{\frac{pq}{p-q}}\, dA \ \le (C\, \|\phi\|_{p,q})^{\frac{pq}{p-q}} \ , \quad \forall\, z \in \mathbb{D}$$

where C *is a positive constant independant of* p *and* q*.*

Lemma 5.7. *Suppose* $z \in \mathbb{D}$ *and* $q \ge 1$*. If there exists* $N > 0$ *such that* $MO_\alpha^q(\phi F)(z) \le N\, ||F||_{BMO_\alpha^q}$ *for all* $F \in VMO_\alpha^q$*, then* $|\phi(w)| \le \ C\ N$ *a.e. on* $B_\alpha(z,t)$*, where* C *is a positive independant of* q*.*

Proof: It is clear that for any $p > q$, we have

$$||F||_{BMO_\alpha^q} \le ||F||_{BMO_\alpha^p} \ , \quad \forall\, F \in BMO_\alpha^p \ .$$

Thus

$$MO_\alpha^q(\phi F)(z) \le N||F||_{BMO_\alpha^p} \ , \quad \forall\, F \in VMO_\alpha^p \ .$$

Therefore by Lemma 5.5, there is a positive constant C independant of p and q such that

$$\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)} \left|\frac{\phi_T}{C\ N}\right|^{\frac{pq}{p-q}} \ dA \ \le \ 1 \ .$$

Letting $p \to q^+$, we have $|\phi(w)| \le \ CN$ *a.e.* on $B_\alpha(z,t)$. $\square$

Corollary 5.8. *Suppose* $p \ge 1$*. If* $\phi BMO_\alpha^p \subset BMO_\alpha^p$*, (or* $\phi VMO_\alpha^p \subset VMO_\alpha^p$ *) then* $\phi \in L^\infty$*. Moreover,* $\|\phi\|_\infty \le \|\phi\|_{p,p}$*.*

Proof: In fact, for any positive integer m, we have $\phi^m VMO_\alpha^p \subset BMO_\alpha^p$, with $\|\phi^m\|_{p,p} \le \|\phi\|_{p,p}^m$. By Lemma 5.7, there exists a constant $C > 0$, such that

$$|\phi(z)| = (|\phi^m(z)|)^{\frac{1}{m}} \le (C\|\phi^m\|_{p,p})^{\frac{1}{m}} \le C^{\frac{1}{m}}\|\phi\|_{p,p} \quad a.e. \ on\ \mathbb{D}.$$

Letting $m \to \infty$, we have the corollary. $\square$

Lemma 5.9. *Suppose* $z \in \mathbb{D}$ *and* $1 \le q \le p < \infty$*. If there exists* $N > 0$ *such that* $MO_\alpha^q(\phi F)(z) \le N||F||_{BMO_\alpha^p}$ *for all* $F \in VMO_\alpha^p$*, then*

$$\frac{1}{(1-|z|)^{\alpha-1}} MO_\alpha^q(\phi)(z) \le CN$$

where C *is a positive constant independant of* p *and* q*.*

Proof: For $F \in VMO_\alpha^p$, we let

$$g = F - \widehat{F}(z) \ .$$

Then

$$\widehat{F}(z)\left(\phi - \widehat{\phi}(z)\right) = \phi F - \widehat{\phi F}(z) - \left(\phi g - \widehat{\phi g}(z)\right) \ .$$

For convenience, denote $r = \frac{pq}{p-q}$ if $p > q$ and $r = \infty$ if $p = q$. Then by Holder Inequality, Lemma 5.5 and Lemma 5.7, we have

$$\begin{aligned} MO_\alpha^q(\phi g)(z) &\leq 2M_\alpha^q(\phi g)(z) \\ &\leq 2M_\alpha^r(\phi)(z)M_\alpha^p(g)(z) \\ &\leq 2(CN)MO_\alpha^p(F)(z) \ . \end{aligned}$$

Hence by the above identity, we have

$$|\widehat{F}(z)|MO_\alpha^q(\phi)(z) \leq MO_\alpha^q(\phi F)(z) + MO_\alpha^q(\phi g)(z) \leq CN||F||_{BMO_\alpha^p} \ .$$

To complete the proof, we need to pick a test function $F_z \in VMO_\alpha^p$ such that $\|F_z\|_{BMO_\alpha^p} \leq C$ and $|\widehat{F_z}(z)| \geq \frac{1}{(1-|z|)^{\alpha-1}}$.

Consider the function $F_z(w) = \frac{1}{(1-\overline{w}\ z)^{\alpha-1}}$. First we verify that $F_z \in VMO_\alpha^p$ by using Theorem 3.1. Since $\nabla F_z(w) = \frac{(\alpha-1)z}{(1-\overline{w}\ z)^{\alpha}} < 1, i >$, we have

$$S(F_z)(w) = (1-|w|)^\alpha \frac{(\alpha-1)|z|\sqrt{2}}{|1-\overline{z}w|^\alpha} \leq (1-|w|)^\alpha \frac{(\alpha-1)|z|\sqrt{2}}{(1-|z|)^\alpha} \to \ 0$$

as $|w| \to 1$. This shows that F_z is in VMO_α^p. Moreover

$$\|F_z\|_{L^1(\mu)} = \int \frac{d\mu}{(1-\overline{w}\ z)^{\alpha-1}} \leq c_\alpha \int \frac{1}{(1-|w|)^{\alpha-1}}\ (1-|w|)^{\alpha-1} dA(w) \leq c_\alpha \ .$$

Therefore, we have

$$\|F_z\|_{BMO_\alpha^p} = \|F_z\|_{L^1(\mu)} + \|S(F_z)\|_\infty \leq c_\alpha + \sqrt{2}\,(\alpha-1) \ .$$

Also, since $F_z(w)$, is analytic in z, we have $F_z(z) = \widehat{F_z}(z)$. Therefore

$$\frac{1}{2^{\alpha-1}(1-|z|)^{\alpha-1}} \leq \frac{1}{(1-|z|^2)^{\alpha-1}} = F_z(z) = \widehat{F_z}(z)$$

and this is enough to conclude the lemma. □

Corollary 5.10. *Suppose* $1 \leq q \leq p < \infty$*. If* $\phi BMO_\alpha^p \subset BMO_\alpha^q$*, then*

$$\frac{1}{(1-|z|)^{\alpha-1}}\ MO_\alpha^q(\phi)(z)\ \leq C\ \|\phi\|_{p,q} \quad \forall\ z \in \mathbb{D}$$

where C *is a positive constant independent of* p *and* q*.*

Proof of Theorem 5.1 :
We prove only the results of multipliers from BMO^p_α to BMO^p_α.

The part (i) of the Theorem is proved as Corollary 5.4.

Also it is clear that the necessary conditions of (ii) and (iii) follow easily from Corollories 5.6, 5.8, 5.10.

To prove the conditions (ii) and (iii) are sufficient, we denote $N_1 = \|\phi\|_\infty$ if $p = q$ and $N_1 = \|M^r_\alpha(\phi)\|_\infty$, with $r = \frac{pq}{p-q}$, if $q < p$; and

$$N_2 = \left\| \frac{1}{(1-|z|)^{\alpha-1}} \; MO^q_\alpha(\phi)(z) \right\|_\infty .$$

For $F \in BMO^p_\alpha$ and any fixed $z \in \mathbb{D}$, let $g = F - \widehat{F}(z)$. Recall the identity in the proof of Lemma 5.9

$$\phi F - \widehat{\phi F}(z) = \phi g - \widehat{\phi g}(z) + \widehat{F}(z)\left(\phi - \widehat{\phi}(z)\right)$$

and the estimate

$$MO^q_\alpha(\phi g)(z) \le 2M^r_\alpha(\phi)(z)\; M^p_\alpha(g)(z) .$$

We have

$$\begin{aligned} MO^q_\alpha(\phi F)(z) &\le MO^q_\alpha(\phi g) + |\widehat{F}(z)| MO^q_\alpha(\phi)(z) \\ &\le 2N_1 MO^p_\alpha(F)(z) + |\widehat{F}(z)| MO^q_\alpha(\phi)(z) . \end{aligned}$$

By Lemma 4.2 we can continue above estimate as follows

$$\begin{aligned} MO^q_\alpha(\phi F)(z) &\le 2N_1 MO^p_\alpha(F)(z) + C||F||_{BMO^p_\alpha}\frac{1}{(1-|z|)^{\alpha-1}} MO^q_\alpha(\phi)(z) \\ &\le C(N_1+N_2)||F||_{BMO^p_\alpha} . \end{aligned} \tag{5.9}$$

To estimate $L^1(\mu)$ norm of ϕF for any $F \in BMO^p_\alpha$, we note that the multiplier ϕ itself is in BMO^q_α and $\|\phi\|_{BMO^q_\alpha} \le (N_1 + N_2)$. Now by Lemma 4.2 we have the following

$$|\widehat{\phi}(z)| \le C\|\phi\|_{BMO^q_\alpha}\frac{1}{(1-|z|)^{\alpha-1}} \le C(N_1+N_2)\frac{1}{(1-|z|)^{\alpha-1}} , \quad \forall \;\; z \in \mathbb{D} .$$

Also, consider the following decomposition

$$\phi F = \phi\left(F - \widehat{F}(z)\right) + \phi \widehat{F}(z) .$$

Then

$$\begin{aligned} M_\alpha^1(\phi F)(z) &\leq M_\alpha^q(\phi F)(z) \\ &\leq M_\alpha^q\left(\phi(F-\widehat{F}(z))\right)+|\widehat{F}(z)|M_\alpha^q(\phi)(z) \\ &\leq M_\alpha^r(\phi)(z)M_\alpha^p\left(F-\widehat{F}(z)\right)(z)+|\widehat{F}(z)|M_\alpha^r(\phi)(z) \\ &= N_1\left(MO_\alpha^p(F)(z)+|\widehat{F}(z)|\right) \\ &\leq CN_1||F||_{BMO_\alpha^p}\frac{1}{(1-|z|)^{\alpha-1}}\ . \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\phi F\|_{L^1(\mu)} &\leq C\|M_\alpha^1\|_{L_1(\mu)} \\ &\leq CN_1||F||_{BMO_\alpha^p}\int\frac{1}{(1-|z|)^{\alpha-1}}d\mu \\ &= CN_1||F||_{BMO_\alpha^p}\ . \end{aligned} \tag{5.10}$$

Now from (5.9) and (5.10), we can conclude that

$$\|\phi F\|_{BMO_\alpha^q}=\|\phi F\|_{L^1(\mu)}+\|MO_\alpha^q(\phi F)\|_\infty\ \leq C(N_1+N_2)||F||_{BMO_\alpha^p}$$

where C is a positive constant independant of p and q. Hence $\phi BMO_\alpha^p\subset BMO_\alpha^q$ if N_1 and N_2 are bounded. i.e. ϕ is a multiplier.

The proof of part (iv) is splited into two cases. First consider the case $p=q$. Then from part (ii) of the theorem, we have $\phi BMO_\alpha^p\subset BMO_\alpha^p$ if and only if $\phi(z)$ and $(1-|z|)|\nabla\phi(z)|$ are bounded. But by Lemma 4.4, we have

$$MO_\alpha^p(\phi)(z)\leq CS(\phi)(z)\ .$$

Therefore, $\phi BMO_\alpha^p\subset BMO_\alpha^p$ if and only if $\phi(z)$ and $(1-|z|)|\nabla\phi(z)|$ are bounded and operator norm equivalent to

$$\|\phi\|_\infty+\sup_{z\in\mathbb{D}}(1-|z|)|\nabla\phi(z)|\ .$$

For $q<p$, from part (iii) of the theorem we have $\phi BMO_\alpha^p\subset BMO_\alpha^q$ if and only if $\left(\frac{1}{|B_\alpha(z,t)|}\int_{B_\alpha(z,t)}|\phi|^r\right)^{\frac{1}{r}}$ and $\frac{1}{(1-|z|)^{\alpha-1}}MO_\alpha^q(\phi)(z)$ are bounded. Therefore again by Lemma 4.4, we have $\phi BMO_\alpha^p\subset BMO_\alpha^q$ if and only if $M_\alpha^r(\phi)(z)$ and $(1-|z|)|\nabla\phi(z)|$

are bounded. Now

$$\begin{aligned} M_\alpha^r(\phi)(z) &\leq |\phi(z)| + M_\alpha^r\left(\phi - \phi(z)\right)(z) \\ &= |\phi(z)| + MO_\alpha^r\left(\phi - \phi(z)\right)(z) \\ &\leq |\phi(z)| + C \sup_{w\in\mathbb{D}} S(\phi)(w)\ . \end{aligned}$$

Hence, $\phi BMO_\alpha^p \subset BMO_\alpha^q$ if and only if $\phi(z)$ and $(1-|z|)|\nabla\phi(z)|$ are bounded and the operator norm is equivalent to

$$\|\phi\|_\infty + \sup_{z\in\mathbb{D}}(1-|z|)|\nabla\phi(z)|\ . \qquad \square$$

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