Math 323 Name: SOLUTIONS
Quiz 3
September 24, 2009 Show your work for full credit.

1. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   Since the columns of
   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
   are linearly independent, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $x_1 = 0$ and $x_2 = 0$.

   Since the columns of
   $$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
   are linearly dependent, the equation $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution; for example, $x_1 = 0$ and $x_2 = 1$.

2. Prove that if $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ contains the zero vector, then it is linearly dependent. You must use the *definition* of linear dependence.

   Reordering if necessary, we suppose $\mathbf{v}_1 = \mathbf{0}$. Then the linear dependence relationship $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ has a nontrivial solution $c_1 = 1, c_2 = 0, \cdots, c_p = 0$.

3. True or False. If S is a linearly dependent set, then each vector in S is a linear combination of the other vectors in S.

   False. Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$. By problem 2, S is linearly dependent; however, $\mathbf{v}_2 \neq c_1\mathbf{v}_1$ for any value of c_1.

4. Let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $\mathbf{y}_1 = (-1, 6)$ and $\mathbf{y}_2 = (2, 5)$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Use the *definition* of linear transformation to find the image of $(5, -3)$. Do not use the standard matrix for T.

   $$T((5, -3)) = T(5\mathbf{e}_1 + (-3)\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5(-1, 6) - 3(2, 5) = (-11, 15)$$

5. Use the *definition* of linear dependence to prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

   Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution $x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3$; therefore, $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0}$ has nontrivial solution $x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3,\ x_4 = 0$ showing that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.