MATH 323
EXAM 2
NOVEMBER 5, 2009

NAME: SOLUTIONS

SHOW YOUR WORK FOR FULL CREDIT.

1. Let $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 \leq 1 \right\}$. Is H a subspace of $\mathbf{R}^2$? Prove or disprove.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in H, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin H$

H IS NOT A SUBSPACE OF $\mathbb{R}^2$

2. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

$AA^{-1} = I$

$\det(AA^{-1}) = \det I$

$\det A \det A^{-1} = 1$

$\det A^{-1} = \dfrac{1}{\det A}$

3. Find a basis for the set of vectors in $\mathbf{R}^2$ on the line $3x_1 + 7x_2 = 0$.

$x_1 = -\frac{7}{3} x_2$

$x = \begin{bmatrix} -\frac{7}{3} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -7/3 \\ 1 \end{bmatrix}$

BASIS IS $\begin{bmatrix} -7/3 \\ 1 \end{bmatrix}$ (OR $\begin{bmatrix} -7 \\ 3 \end{bmatrix}$)

4. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Prove that $B = C$.

$(B - C)D = 0$

$(B - C)DD^{-1} = 0D^{-1}$

$(B - C)I = 0$

$B - C = 0$

$B = C$

5. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the columns of A are linearly independent. Hint: Consider the equation $A\mathbf{x} = \mathbf{0}$.

$$Ax = 0$$
$$CAx = C0$$
$$Ix = 0$$
$$x = 0$$

SINCE $Ax=0$ HAS ONLY THE TRIVIAL SOLUTION, THE COLUMNS OF A ARE LINEARLY INDEPENDENT.

6. Using the definition of linear independence, prove that the subset $\{1+t, 1+t^2, t+t^2\}$ of $\mathbf{P}_2$ is linearly independent.

$$c_1(1+t) + c_2(1+t^2) + c_3(t+t^2) = 0$$
$$(c_1+c_2) + (c_1+c_3)t + (c_2+c_3)t^2 = 0$$

$$\left.\begin{aligned} c_1+c_2 &= 0 \\ c_1+c_3 &= 0 \end{aligned}\right\} \longrightarrow c_2 - c_3 = 0$$
$$c_2+c_3 = 0 \qquad c_2 + c_3 = 0$$
$$c_2 = 0$$
$$c_1 = -c_2$$

$$c_1 = 0$$
$$c_3 = -c_1$$
$$c_3 = 0$$

7. Let F be a fixed $m \times n$ matrix, and let H be the set of all matrices A in $M_{n \times p}$ with the property that $FA = 0$ (the zero matrix in $M_{m \times p}$). Show that H is a subspace of $M_{n \times p}$.

(1) $F0 = 0 \Rightarrow 0 \in H$

(2) $A, B \in H$, $F(A+B) = FA + FB = 0 + 0 = 0$
$\Rightarrow A + B \in H$

(3) $A \in H$, c ANY SCALAR,
$F(cA) = cFA = c0 = 0 \Rightarrow cA \in H$

8. Let A and B be $n \times n$ matrices. Prove that if AB is invertible, then so is B.

AB INVERTIBLE, SO WE HAVE C SUCH THAT

$C(AB) = I$

$(CA)B = I$

BY THE I.M.T., B IS INVERTIBLE.

9. Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a subset of V. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent in V, then the set of images $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly independent in W.

$c_1 v_1 + \cdots + c_p v_p = 0$, NOT ALL $c_j = 0$.

$T(c_1 v_1 + \cdots + c_p v_p) = T(0)$, NOT ALL $c_j = 0$

$c_1 T(v_1) + \cdots + c_p T(v_p) = 0$, NOT ALL $c_j = 0$

THEREFORE, $\{T(v_1), \ldots, T(v_p)\}$ LIN. DEP. IN W.

10. Suppose that $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 1$. Find the value.

$$\begin{vmatrix} a & b & c & d \\ 3e & 3f & 3g & 3h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = 3 \qquad \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 5e+m & 5f+n & 5g+o & 5h+p \end{vmatrix} = 1$$

$$\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix} = 1 \qquad \begin{vmatrix} e & f & g & h \\ a & b & c & d \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = -1$$

$$\begin{vmatrix} a & b & c & d \\ 2e & 2f & 2g & 2h \\ 3i & 3j & 3k & 3l \\ 4m & 4n & 4o & 4p \end{vmatrix} = 24$$