MATH 323
EXAM 1
OCTOBER 1, 2009

NAME: SOLUTIONS

SHOW YOUR WORK FOR FULL CREDIT.

1. True or False. A consistent linear system has a unique solution. Justify your answer.

FALSE, THE SYSTEM COULD HAVE INFINITELY MANY SOLUTIONS.

2. True or False. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix. Justify your answer.

TRUE, BY DEFINITION OF A BASIC VARIABLE.

3. Write a vector equation that is equivalent to the system of equations

$$\begin{aligned} x_2 + 5x_3 &= 9 \\ 4x_1 + 6x_2 - x_3 &= 2 \\ -x_1 + 3x_2 - 8x_3 &= 15 \end{aligned}$$

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

4. Fill in the blank. The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is A LINEAR COMBINATIONS OF THE COLUMNS OF A.

5. Write the general solution of $10x_1 - 3x_2 - 2x_3 = 0$ in parametric vector form.

$$x_1 = \frac{3}{10}x_2 + \frac{2}{10}x_3$$

$$x = x_2 \begin{bmatrix} 3/10 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2/10 \\ 1 \end{bmatrix}$$

6. Prove that if $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ contains the zero vector, then it is linearly dependent. You must use the *definition* of linear dependence.

SUPPOSE $v_1 = 0$. THEN THE LINEAR DEPENDENCE RELATIONSHIP $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ HAS NONTRIVIAL SOLUTION

$$c_1 = 1,\ c_2 = 0,\ \ldots,\ c_p = 0.$$

7. Suppose $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{w} = \mathbf{p} + \mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Do not skip any steps.

$$Aw = A(p+q) = Ap + Aq = b + 0 = b.$$

8. True or False. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent. Justify your answer.

FALSE, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ ARE LINEARLY DEPENDENT.

9. Let $T(x_1, x_1) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$ be a linear transformation mapping $\mathbf{R}^2$ to $\mathbf{R}^3$.

(a) Write down the standard matrix A for T.

$T(1,0) = (3,5,1)$, $T(0,1) = (1,7,3)$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

(b) Is T one-to-one? Explain using the matrix A.

YES, THE COLUMNS OF A ARE LINEARLY INDEPENDENT (THEY ARE NOT SCALAR MULTIPLES OF EACH OTHER).

(c) Is T onto? Explain using the matrix A.

NO, THE COLUMNS OF A DO NOT SPAN $\mathbb{R}^3$ (IT TAKES AT LEAST THREE VECTORS TO SPAN $\mathbb{R}^3$).

10. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^2$, compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$.

$$u^T v = [u_1 \ u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

$$v^T u = [v_1 \ v_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = v_1 u_1 + v_2 u_2$$

$$u v^T = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} [v_1 \ v_2] = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{bmatrix}$$

$$v u^T = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} [u_1 \ u_2] = \begin{bmatrix} v_1 u_1 & v_1 u_2 \\ v_2 u_1 & v_2 u_2 \end{bmatrix}$$