Differential Equations Review
Fundamentals of Engineering Examination

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SECTION I

Equations of Order One

1. *Separation of Variables:* Consider the equation $M(x, y)\,dx + N(x, y)\,dy = 0$. If this equation can be put in the form $f(x)\,dx = g(y)\,dy$, we say that the variables can be separated. Then, the solution may be obtained by integrating both sides.

   1.1. Solve the equation $\frac{dy}{dx} = \frac{y^2}{1-x}$. ANSWER: $y = \frac{1}{\ln\left|C(1-x)\right|}$, where C is a constant of integration.

   1.2. Solve the equation $\frac{dV}{dP} = -\frac{V}{P}$. ANSWER: $V = C/P$, where C is a constant of integration.

   1.3. Solve the equation $\frac{dx}{dt} = \frac{\csc^2 x}{1+t^2}$. ANSWER: $x - \sin x \cos x = 2\tan^{-1} t + C$, where C is a constant of integration.

2. *Homogeneous Coefficients:* Consider the equation $M(x, y)\,dx + N(x, y)\,dy = 0$. If this equation can be put in the form $y' = f(y/x)$, then making the substitution $v = y/x$ will yield a separable equation.

   REMARK: In certain cases it may be easier to set $v = x/y$ instead.

   2.1. Solve the equation $\left(x^2 - xy + y^2\right)dx - xy\,dy = 0$. ANSWER: $(y - x)\exp(y/x) = C$, where C is a constant of integration.

3. *Exact Equations:* If a function $F(x, y)$ exists such that $dF = M\,dx + N\,dy = 0$, we call the equation $M\,dx + N\,dy = 0$ exact. Recall that $dF = (\partial F/\partial x)\,dx + (\partial F/\partial y)\,dy$. Therefore, if the equation $M\,dx + N\,dy = 0$ is exact, then $M = \partial F/\partial x$ and $N = \partial F/\partial y$.

   THEOREM: If M, N, $\partial M/\partial y$, and $\partial N/\partial x$ are continuous on a simply connected open set, then $M\,dx + N\,dy = 0$ is exact if and only if $\partial M/\partial y = \partial N/\partial x$.

   The solution to an exact equation is found by "partial integration" and is given by $F(x, y) = C$, where C is a constant of integration.

3.1. Find the solution of the equation $3x(xy-2)\,dx+\left(x^3+2y\right)dy=0$, which passes through the point $(1,0)$. ANSWER: $x^3y-3x^2+y^2+3=0$.

4. *Linear Equations:* A first-order differential equation of the form $\frac{dy}{dx}+P(x)\,y=Q(x)$ is called linear. This equation may be solved by multiplying both sides by the integrating factor $\mu(x)=\exp\left(\int P(x)\,dx\right)$. This causes the left side of the equation to become $\frac{d}{dx}(\mu\cdot y)$, i.e. exact.

WARNING: The coefficient of y' must be 1.

4.1. Solve the equation $2\left(y-4x^2\right)dx+x\,dy=0$. ANSWER: $y=2x^2+C\,x^{-2}$, where C is a constant of integration.

Equations of Order Two

5. *Equations of the form* $d^2y/dx^2=f\,(x,dy/dx)$*:* Make the substitution $dy/dx=v$ (and therefore $d^2y/dx^2=v'$) in order to reduce the equation to order one.

5.1. Solve the equation $y''=5(y')^2$. ANSWER: $y=-\frac{1}{5}\ln\left|C_1+5x\right|+C_2$, where C_1 and C_2 are constants of integration.

5.2. Solve the equation $xy''=y'+x^5$ subject to the conditions that when $x=1$ we have $y=1/2,\ y'=1$. ANSWER: $24y=x^6+9x^2+2$.

6. *Equations of the form* $d^2y/dx^2=f\,(y,dy/dx)$*:* Make the substitution $dy/dx=v$ (and therefore $d^2y/dx^2=v\cdot dv/dy$) in order to reduce the equation to order one.

REMARK: The fact that $d^2y/dx^2=v\cdot dv/dy$ comes from the chain rule, specifically $d^2y/dx^2=dv/dx=dv/dy\cdot dy/dx=dv/dy\cdot v$.

6.1. Solve the equation $yy''+(y')^2=0$. ANSWER: $y^2=C_1\,x+C_2$, where C_1 and C_2 are constants of integration.

7. *Linear equations with constant coefficients,* $ay''+by'+cy=0$: Finding the roots r_1 and r_2 of the auxiliary equation $ar^2+br+c=0$ leads to the solution as follows.

Case 1: $r_1\neq r_2$ both real $\Rightarrow y=C_1\,e^{r_1x}+C_2\,e^{r_2x}$, where C_1 and C_2 are constants of integration.

Case 2: $r_1=r_2\Rightarrow y=C_1\,e^{r_1x}+C_2\,xe^{r_1x}$, where C_1 and C_2 are constants of integration.

Case 3: $r_1=a+bi,\ r_2=a-bi\Rightarrow y=C_1\,e^{ax}\cos bx+C_2e^{ax}\sin bx$, where C_1 and C_2 are constants of integration.

REMARK: This method may be extended to linear differential equations with constant coefficients whose order is higher than two.

7.1. Solve the equation $y'' - 6y' + 10y = 0$. ANSWER: $y = C_1 e^{3x} \cos x + C_2 e^{3x} \sin x$, where C_1 and C_2 are constants of integration.

7.2. Solve the equation $y'' + 4y' + 4y = 0$ subject to the initial conditions $y(0) = 1, y'(0) = -1$. ANSWER: $y = (1 + x) e^{-2x}$.

8. *Method of Undetermined Coefficients:* The solution of the linear differential equation with constant coefficients $ay'' + by' + cy = f(x)$ is $y = y_h + y_p$ where y_h is the general solution to the associated homogeneous equation $ay'' + by' + cy = 0$ and y_p is any particular solution of $ay'' + by' + cy = f(x)$. The method of undetermined coefficients is based on making a judicious guess as to the form of y_p.

8.1. Solve the equation $y'' + y' - 2y = 2x - 40 \cos 2x$.
ANSWER: $y = C_1 e^{x} + C_2 e^{-2x} - \frac{1}{2} - x + 6 \cos 2x - 2 \sin 2x$, where C_1 and C_2 are constants of integration.

8.2. Solve the equation $y'' + 4y = 8e^{2x} + 12x$.
ANSWER: $y = C_1 \cos 2x + C_2 \sin 2x + 3x + e^{2x}$, where C_1, and C_2 are constants of integration.

9. *Reduction of Order:* Consider the linear differential equation of order two $y'' + P(x) y' + Q(x) y = 0$. If we know one solution $y = y_1$, then a second linearly independent solution $y = y_2$ is given by

$$y_2 = y_1 \cdot \int \frac{\exp\left(-\int P(x)\, dx\right)}{\left[y_1(x)\right]^2} dx .$$

WARNING: The coefficient of y'' must be 1.

9.1. Verify that $y = x$ is one solution of the Legendre differential equation $\left(1 - x^2\right) y'' - 2xy' + 2y = 0$. Find a second solution.

ANSWER: $y = -2 + x \ln \left| \dfrac{1 + x}{1 - x} \right|$.

10. *Variation of Parameters:* Consider the linear differential equation of order two $y'' + P(x) y' + Q(x) y = R(x)$. If we know two linearly independent solutions y_1 and y_2 of the corresponding homogeneous equation $y'' + P(x) y' + Q(x) y = 0$, then a particular solution of $y'' + P(x) y' + Q(x) y = R(x)$ is given by $y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$ where

$$v_1(x) = \int \frac{-R(x)\,y_2(x)}{W(x)}\,dx \quad \text{and} \quad v_2(x) = \int \frac{R(x)\,y_1(x)}{W(x)}\,dx$$

Note: $W(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)$ is the Wronskian of y_1 and y_2.

10.1. Solve the equation $y'' - 3y' + 2y = 1/\left(1 + e^{-x}\right)$.

ANSWER: $y = C_1 e^x + C_2 e^{2x} + \left(e^x + e^{2x}\right)\ln\left(1 + e^{-x}\right)$, where C_1 and C_2 are constants of integration.

11. *Laplace Transforms:* The Laplace transform of f is defined by

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt\,.$$

The Laplace transform obeys the linearity property

$$\mathcal{L}\{af(t) + bg(t)\}(s) = aF(s) + bG(s)$$

where a and b are constants.

Some basic formulas for the Laplace transform are:

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}},\ n = 1, 2, 3, \ \ldots$$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}},\ n = 1, 2, 3, \ \ldots$$

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2 + k^2}$$

$$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}\{e^{at}\sin kt\}(s) = \frac{k}{(s-a)^2 + k^2}$$

$$\mathcal{L}\{e^{at}\cos kt\}(s) = \frac{(s-a)}{(s-a)^2 + k^2}$$

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$$

$$\mathcal{L}\{f''\}(s) = s^2\,\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$

Solving an initial value problem using the method of Laplace transforms.

i) Take the Laplace transform of both sides of the equation.

ii) Obtain and solve an (algebraic) equation for the Laplace transform of the solution.

iii) Determine the solution of the initial value problem by inverting the Laplace transform in part ii).

11.1. Solve the initial value problem $y'(t) + 3y(t) = 13\sin 2t;\ y(0) = 6$.

ANSWER: $y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$.

REMARK: This problem could have been solved by the method of item 7.

11.2. Solve the initial value problem $y''(t)+2y'(t)+y(t)=3t\,e^{-t};\ y(0)=4,\ y'(0)=2$.

ANSWER: $y(t)=\left(4+6t+\frac{1}{2}t^3\right)e^{-t}$.

REMARK: This problem could have been solved by the method of item 8 or item 10.

12. *Cauchy-Euler Equation:* The linear differential equation of order two

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0,$$

where a, b, and c are constants, may be solved by substituting $y=x^r$ into the equation and solving for r.

12.1. Solve the equation $2x^2y''+xy'-y=0$. ANSWER: $y=C_1\,x+C_2\,x^{-1/2}$, where C_1 and C_2 are constants of integration.

SECTION II

Problems for further review.

1. $xy^3\,dx+e^{x^2}\,dy=0$. ANSWER: $e^{-x^2}+y^{-2}=C$.

2. $x\cos^2 y\,dx+\tan y\,dy=0$. ANSWER: $x^2+\tan^2 y=C^2$.

3. $x^2y'=4x^2+7xy+2y^2$. ANSWER: $x^2(y+2x)=C(y+x)$.

4. $y(9x-2y)\,dx-x(6x-y)\,dy=0$; when $x=1,\ y=1$. ANSWER: $3x^3-x^2y-2y^2=0$.

5. $\left(1+y^2+xy^2\right)dx+\left(x^2y+y+2xy\right)dy=0$. ANSWER: $2x+y^2(1+x)^2=C$.

6. $(r+\sin\theta-\cos\theta)\,dr+r(\sin\theta+\cos\theta)\,d\theta=0$. ANSWER: $r^2+2r(\sin\theta-\cos\theta)=C$.

7. $y'=x-2xy$. Solve by two different methods. ANSWER: $2y=1+C\,e^{-x^2}$.

8. $(2x+3)\,y'=y+(2x+3)^{1/2}$; when $x=-1,\ y=0$. ANSWER: $2y=(2x+3)^{1/2}\ln(2x+3)$.

9. $y^2y''+(y')^3=0$. ANSWER: $x=C_1\,y-\ln\left|C_2\,y\right|$.

10. $y''\cos x=y'$. ANSWER: $y=C_2+C_1\ln(1-\sin x)$.

11. $y''=2y(y')^3$. ANSWER: $y^3=3(C_2-x-C_1\,y)$.

12. $y''=-e^{-2y}$; when $x=3,\ y=0,\ y'=-1$. ANSWER: $y=\ln(4-x)$.

13. $y''+6y'+9=0$. ANSWER: $y=(C_1+C_2\,x)e^{-3x}$.

14. $2y^{(4)}-3y'''-2y''=0$. ANSWER: $y=C_1+C_2\,x+C_3\,e^{2x}+C_4\,e^{-x/2}$.

15. $y'' - 4y' + 7y = 0$. ANSWER: $y = C_1 e^{2x} \cos\sqrt{3}x + C_2 e^{2x} \sin\sqrt{3}x$.

16. $y^{(4)} + 2y''' + 10y'' = 0$. ANSWER: $y = C_1 + C_2 x + C_3 e^{-x} \cos 3x + C_4 e^{-x} \sin 3x$.

17. $y^{(4)} + 18y'' + 81y = 0$. ANSWER: $y = (C_1 + C_2 x)\cos 3x + (C_3 + C_4 x)\sin 3x$.

18. $y'' + y' = \sin x$. ANSWER: $y = C_1 + C_2 e^{-x} - \frac{1}{2}\sin x - \frac{1}{2}\cos x$.

19. $y'' - 3y' - 4y = 5e^{4x}$. ANSWER: $y = (C_1 + x)e^{4x} + C_2 e^{-x}$.

20. $y'' + 3y' = -18x$; when $x = 0, y = 0, y' = 5$. ANSWER: $y = 1 + 2x - 3x^2 - e^{-3x}$.

21. $y'' + 2y' + y = (e^x - 1)^{-2}$. ANSWER: $y = e^{-x}\left(C_1 + C_2 x - \ln\left|1 - e^{-x}\right|\right)$.

22. Verify that $y = e^x$ is a solution of $(x-1)y'' - xy' + y = 0$. Find the general solution of $(x-1)y'' - xy' + y = 1$. ANSWER: $y = C_1 x + C_2 e^x + 1$.

23. $y'' + y = \sec^3 x$. Solve by two different methods.
ANSWER: $y = C_1 \cos x + C_2 \sin x + \frac{1}{2}\sec x$.

24. $y''(t) - 3y'(t) + 2y(t) = e^{3t}; y(0) = 0, y'(0) = 0$. Solve by two different methods.
ANSWER: $y = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$.

25. $y''(t) - 4y'(t) + 4y(t) = 4\cos 2t; y(0) = 2, y'(0) = 5$. Solve by two different methods.
ANSWER: $y = 2e^{2t}(1+t) - \frac{1}{2}\sin 2t$.

26. $x^2 y'' + 2xy' - 12y = 0$. ANSWER: $y = C_1 x^3 + C_2 x^{-4}$.

27. $x^2 y'' - 3xy' + 4y = 0$. ANSWER: $y = x^2(C_1 + C_2 \ln x)$.

28. *Newton's Law of Cooling:* Let $T(t)$ represent the temperature of an object at time t and suppose that this object is placed in an environment where the ambient temperature at time t is given by $M(t)$. Experimental evidence shows that a good model of this situation is given by the equation $dT/dt = k\left(M(t) - T(t)\right)$ where k is a constant of proportionality.

    28.1. Suppose that a thermometer, which has been at the reading 70°F inside a house, is placed outside where the air temperature is 10°F. Three minutes later it is found that the thermometer reading is 25°F. Show that $T(t) = 10 + 60\exp\left(-\frac{1}{3}t\ln 4\right)$.

29. *Mechanical Vibrations:* Suppose a weight with mass m is suspended from a spring with spring constant k. Let $x(t)$ represent the distance of the weight from equilibrium at time t where downward is taken as the positive direction. This situation can be modeled

by the differential equation $mx''(t)+bx'(t)+kx(t)=f(t)$ where b is a damping constant and $f(t)$ is a forcing function.

REMARK: Recall that *Hooke's Law* states the following. A spring stretched from its rest position exerts a restoring force $\mathbf{F}$ whose magnitude is $F=ks$, where s is the amount of elongation of the spring and $k>0$ is the spring constant.

29.1. *Undamped Vibrations* ($b=0$): The differential equation becomes $x''(t)+\beta^2 x(t)=f(t)$ where $\beta^2=k/m$. Notice that a factor of $1/m$ has been absorbed into the forcing function.

29.1.1. Suppose that a spring is such that it would be stretched 6 inches by a 12-lb. weight. Let the weight be attached to the spring and pulled down 4 inches below the equilibrium point. If the weight is started with an upward velocity of 2 ft/sec, describe the motion. Assume that no damping or forcing function is present. Show that $x(t)=-\frac{1}{4}\sin 8t+\frac{1}{3}\cos 8t$.

29.1.2. Suppose that the forcing function in 29.1 is given by $f(t)=A\sin\omega t$ where $\omega\neq\beta$. Assuming the initial conditions are $x(0)=x_0$ and $x'(0)=v_0$, show that

$$x(t)=\frac{v_0}{\beta}\sin\beta t+x_0\cos\beta t-\frac{A\omega}{\beta\left(\beta^2-\omega^2\right)}\sin\beta t+\frac{A}{\beta^2-\omega^2}\cos\beta t.$$

REMARK: The case where $A=0$ is exactly the case where no forcing function is present. Compare the solution of 29.1.2, with $A=0$, to the solution of 29.1.1.

29.1.3. Suppose that the forcing function in 29.1 is given by $f(t)=A\sin\omega t$ where $\omega=\beta$ (this is the case of resonance). Assuming the initial conditions are $x(0)=x_0$ and $x'(0)=v_0$, show that

$$x(t)=\frac{v_0}{\beta}\sin\beta t+x_0\cos\beta t+\frac{A}{2\beta^2}\left(\sin\beta t-\beta t\cos\beta t\right).$$

29.2. *Damped Vibrations:* The differential equation becomes $x''(t)+2\gamma x'(t)+\beta^2 x(t)=f(t)$ where $2\gamma=b/m$. Assume the initial conditions are $x(0)=x_0$, $x'(0)=v_0$ and there is no forcing function.

29.2.1. *Underdamped* (*or Oscillatory*) *Motion:* This is the case where $\beta>\gamma$. Show that $x(t)=e^{-\gamma t}\left(c_1\cos\delta t+c_2\sin\delta t\right)$ where $\delta^2=\beta^2-\gamma^2$, $\delta>0$.

29.2.2. *Critically Damped Motion:* This is the case where $\beta=\gamma$. Show that $x(t)=e^{-\gamma t}\left(c_1+c_2 t\right)$.

29.2.3. *Overdamped Motion:* This is the case where $\beta < \gamma$. Show that

$$x(t) = c_1 e^{(-\gamma+\sigma)t} + c_2 e^{(-\gamma-\sigma)t} \text{ where } \sigma^2 = \gamma^2 - \beta^2,\ \sigma > 0.$$

REMARK: It is very instructive to choose specific values for m, b, k, x_0, v_0, and to compare the graphs of x versus t in the cases where motion is underdamped, critically damped, and overdamped.