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1 Results

My research plan was to study the geometry of cubic polynomials. After an exhaustive study of the literature, I was able to answer the question that was posed. Starting with a given critical point (in a specified region of the unit disk) I was able to reconstruct the associated unique polynomial (in a special class of cubic polynomials) both geometrically and analytically. After this, I coauthored a research paper entitled “More Geometry of Cubic Polynomials; An Inverse Problem”. This paper follows, beginning with Section 2 and ending with references. This paper has been submitted to Math Magazine. I also presented this research on two different occasions. As part of the UW-Platteville Mathematics Colloquium Series, Spring 2015, I gave a talk to students and faculty on campus titled “More Geometry of Cubic Polynomials”. Additionally, these results were presented at the Wisconsin Section Meeting of the Mathematical Association of America in Spring 2015. I am pleased to report that this research has significantly expanded my knowledge in a fundamental area of mathematics that will hopefully lead to future undergraduate research projects.

2 More Geometry of Cubic Polynomials; An Inverse Problem

Several recent papers ([1], [2], [3]) have studied the geometry of cubic polynomials, specifically asking, how the critical points of a cubic polynomial depend upon its roots. Frayer, Kwon, Schafhauser, and Swenson [1] studied the critical points of a family of polynomials

\[ \Gamma = \{ q : \mathbb{C} \to \mathbb{C} \mid q(z) = (z - 1)(z - r_1)(z - r_2), |r_1| = |r_2| = 1 \} . \]

They show that almost every \( c \) in a specified portion of the unit disk is the critical point of a unique \( p \in \Gamma \). The purpose of this note is to reconstruct the unique polynomial from a given critical point.

Circles which are internally tangent to the unit circle at 1 will play an important role in what follows. Given \( \alpha > 0 \), denote by \( T_\alpha \) the circle of diameter \( \alpha \) passing through 1 and \( 1 - \alpha \) in the complex plane. That is,

\[ T_\alpha = \left\{ z \in \mathbb{C} : \left| z - \left( 1 - \frac{\alpha}{2} \right) \right| = \frac{\alpha}{2} \right\} . \]
Each $z \neq 1$ in the unit disk lies on a unique $T_\alpha$ with $0 < \alpha \leq 2$. For such a $z$, the perpendicular bisector of $\overline{1z}$ intersects the real axis at $1 - \frac{\alpha}{2}$ and it follows that $\alpha$ is constructible. See Figure 1.

**Lemma 1.** [1] Let $z \in \mathbb{C}$ with $\text{Re}(z) < 1$. We have $z \in T_\alpha$ if and only if
\[
\frac{1}{\alpha} = \text{Re}\left(\frac{1}{1-z}\right).
\]

For $c \in \mathbb{C}$ the main results of [1] include:

- If $c \notin \{1, -\frac{1}{3}\}$ lies on $T_\alpha$ for some $\alpha \in \left[\frac{2}{3}, 2\right]$, then there is a unique $p \in \Gamma$ with a critical point at $c$.
- If $c$ lies strictly inside $T_{2/3}$, or strictly outside $T_2$, then there is no $p \in \Gamma$ with a critical point at $c$.
- $p \in \Gamma$ has a critical point at 1 if and only if $p(z) = (z-1)^2(z-r)$ for some $r$ on the unit circle.
- $p \in \Gamma$ has a critical point at $-\frac{1}{3}$ if and only if $p(z) = (z-1)(z-r)\left(z + \frac{5r+3}{3r-5}\right)$ for some $r$ on the unit circle.

Additionally, for $c \in T_{2/3} \cup T_1 \cup T_{4/3} \cup T_2$, [1] reconstructs the unique $p \in \Gamma$ with a critical point at $c$. The question remains, given $c \in \Delta$ with
\[
\Delta = \{z \mid z \in T_\alpha \text{ with } \alpha \in \left(\frac{2}{3}, 1\right) \cup \left(1, \frac{4}{3}\right) \cup \left(\frac{4}{3}, 3\right)\},
\]
what is the unique $p \in \Gamma$ with critical point at $c$? In addition to representing $p(z)$ analytically, we will construct the roots of $p(z)$ geometrically [2].

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[1] In [1], an eloquent geometric construction is used to determine which $z \in T_2$ can be a zero of $p''(z)$ for some $p \in \Gamma$. 

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\section{Results}

Suppose \( p \in \Gamma \) with roots \( r_1, r_2, \) and 1, and let \( c \in \Delta \) be a critical point of \( p \). For the following analysis it is useful to note

\[ c \notin T_1 \iff \left| \frac{1}{2} - c \right| \neq \frac{1}{2} \iff |2c - 1| \neq 1. \tag{1} \]

Direct calculations give

\[ 0 = p'(c) = 3c^2 - 2c(r_1 + r_2) + (r_1 + r_2 + r_1r_2). \]

When \( c \notin T_1 \), we have \( r_1 \neq 2c - 1 \), and

\[ r_2 = \frac{(2c - 1)r_1 + (2c - 3c^2)}{r_1 + (1 - 2c)}. \]

We define

\[ f_c(z) = \frac{(2c - 1)z + (2c - 3c^2)}{z + (1 - 2c)} \]

and let \( S_c \) denote the image of the unit circle under \( f_c \). That is, \( S_c = f_c(T_2) \) and \( f_c(r_1) = r_2 \). Since \( c \notin T_1 \) and \( f_c \) is a linear fractional transformation, it follows from equation \( (1) \) that \( S_c \) is a circle. When \( c \neq 1 \), we have \((f_c)^{-1} = f_c \) and \( f_c \) maps \( T_2 \) onto \( S_c \) and \( S_c \) onto \( T_2 \). Hence \( f_c \) restricts to a one-to-one correspondence from \( S_c \cap T_2 \) to itself, and if \( c \) is a critical point of \( p \), we have \( \{r_1, r_2\} \subseteq S_c \cap T_2 \).

Given \( c \) in the unit disk, we define \( L \) to be the line through 1 and \( c \). Line \( L \) intersects \( T_2 \) at \( 1 \), and a second point which we will denote by \( w \). See Figure 1. As \( f_c \) is a linear fractional transformation, \( f_c(L) \) is a line if there exists \( z \in L \) with \( z + (1 - 2c) = 0 \). As \( f_c(c) = c \) and direct calculations give \(|1 - c| + |c - (2c - 1)| = |1 - (2c - 1)| \), it follows that \( z = 2c - 1 \in L \) and \( f_c(L) \) is a line. Additionally, \( f_c(1) = \frac{3c - 1}{2} \) and \( L \) is a line through 1 and \( c \). Direct calculations give \(|1 - c| + |c - \frac{3c - 1}{2}| = |1 - \frac{3c - 1}{2}| \) and it follows that \( f_c(1) \in L \), and hence \( f_c(L) = L \).

\begin{lemma}
For \( c \in \Delta \), exactly one of \( f_c(1) \) and \( f_c(w) \) lie inside \( T_2 \).
\end{lemma}

\begin{proof}
Since \( f_c(1) = \frac{3c - 1}{2} \),

\[ \frac{1}{1 - f_c(1)} = \frac{2}{3} \frac{1}{1 - c} \]

and Lemma 1 implies that \( f_c(1) \in T_{2\alpha} \). Direction calculations give \( w = \frac{2}{\alpha} (c - 1) + 1 \) and \( f_c(w) = \frac{2 - 3\alpha}{2 - 2\alpha} (c - 1) + 1 \). Therefore,

\[ \frac{1}{1 - f_c(w)} = \left( \frac{2 - 2\alpha}{4 - 3\alpha} \right) \frac{1}{1 - c} \]

and Lemma 1 implies that \( f_c(w) \in T_{\frac{4\alpha - 3\alpha^2}{2 - 2\alpha}} \). Comparing the graphs of \( y_1 = \frac{3\alpha}{2} \) and \( y_2 = \frac{4\alpha - 3\alpha^2}{2 - 2\alpha} \) in Figure 2 shows that when \( \alpha \in \left( \frac{2}{3}, 1 \right) \cup \left( 1, \frac{4}{3} \right) \cup \left( \frac{4}{3}, 2 \right) \), exactly one of \( f_c(1) \) and \( f_c(w) \) lie inside \( T_2 \). \end{proof}

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Figure 2: $f_c(1)$ versus $f_c(w)$.

Since $\left\{ f_c(1), f_c(w) \right\} \subseteq S_c$, Lemma 2 implies that $|S_c \cap T_2| = 2$. Specifically, as $\left\{ r_1, r_2 \right\} \subseteq S_c \cap T_2$ and $c \notin T_2$, we have $S_c \cap T_2 = \{ r_1, r_2 \}$ with $r_1 \neq r_2$.

We will now study some basic properties of $S_c$. By definition, $z \in S_c$ if and only if there exists $w \in T_2$ with $f_c(w) = z$. That is, $f_c^{-1}(z) \in T_2$ if and only if $|f_c^{-1}(z)| = |f_c(z)| = 1$, or equivalently,

$$\frac{(2c-1)z + (2c-3c^2)}{z + (1-2c)} = 1.$$  

Therefore, $z \in S_c$ if and only if

$$|z - (2c-1)| = |2c-1| \cdot \left| z - \frac{3c^2 - 2c}{2c-1} \right|. \quad (2)$$

It follows from introductory complex analysis that for $k \neq 1$, the solution set of

$$|z - u| = k|z - v| \quad (3)$$

is a circle with center $C = v + \frac{u-v}{k^2-1}$ and radius given by $R^2 = |C|^2 - \frac{k^2|u|^2-|v|^2}{k^2-1}$. Letting $W = z - \frac{3c^2 - 2c}{2c-1}$ in (2), yields

$$\left| W - c - \frac{1}{2} \right| = |2c-1| \left| W - \frac{c - 1}{2(2c-1)} \right|. \quad (4)$$

When $c \notin T_1$, equation (1) gives $|2c-1| \neq 1$. Therefore, equation (3) implies that
the solution set of \(4\) is a circle with center

\[
C = \frac{c - 1}{2(2c - 1)} + \frac{c - 1}{2|2c - 1|^2 - 1} \frac{2}{2c - 1}
\]

\[
= \frac{|c - 1|^2}{2|c - 1|^2 - 1}
\]

\[
= \frac{1}{|2 - \frac{1}{c}|^2 - |\frac{1}{1 - c}|^2}
\]

\[
= \frac{1}{4(1 - \frac{1}{c})}
\]

where the last step follows from Lemma 4. Additionally,

\[
R^2 = |C|^2 - \frac{|2c - 1|^2}{2|2c - 1|^2} \frac{c - 1}{2(2c - 1)} \frac{2}{2c - 1} = |C|^2
\]

so that \(R = |C|\). Since \(z = W + \frac{3c-1}{2}, \) for \(c \notin T_1, S_c\) is a circle with center \(\gamma = \frac{3c-1}{2} + \frac{\alpha}{4\alpha-4}\) and radius \(R = \frac{2}{4\alpha-4}\). Therefore \(f_c(1) = \frac{3c-1}{2}\) and \(\gamma\) have identical imaginary parts.

**Lemma 3.** For \(c \in T_\alpha\) with \(\alpha \in \left(\frac{2}{3}, 1\right) \cup (1, 2), Im(f_c(1)) = Im(\gamma)\).

We are now ready for the main results.

**Property 1.** Let \(c \in \Delta\). The roots of the unique \(p \in \Gamma\) with \(p'(c) = 0\) can be constructed geometrically.

**Proof.** To begin, construct the number \(\alpha\) and points \(A = f_c(1) = \frac{3c-1}{2}\) and \(B = f_c(w) = \frac{3c-1}{2} + \frac{\alpha}{4\alpha-4}\). Line segment \(L_1 = \overline{AB}\) lies on line \(L\) through 1 and \(c\), and Lemma 2 implies that \(L_1\) intersects \(T_2\). See Figure 3.

Construct the perpendicular bisector of \(L_1\) and call this line \(L_2\). As \(L_1\) is a chord of \(S_c\), line \(L_2\) passes through the center of \(S_c\). Construct line \(L_3\) through \(A\) and parallel to the real axis. Lemma 2 implies that \(L_3\) also passes through the center of \(S_c\). Let \(\gamma\) be the intersection of lines \(L_2\) and \(L_3\). Now, construct \(S_c\), the circle with center \(\gamma\) passing through \(f_c(1)\) (and \(f_c(w)\)). As \(L_1\) intersects \(T_2\), the circle \(S_c\) intersects \(T_2\) twice; call these intersections \(r_1\) and \(r_2\). Then \(p(z) = (z - 1)(z - r_1)(z - r_2)\) is the unique polynomial in \(\Gamma\) with \(p'(c) = 0\).

The construction also works for \(c \in T_2 \cup T_4 \cup T_2\), but we cannot guarantee that \(|S_c \cap T_2| = 2\). For such a \(c\), line segment \(L_1\) will intersect \(T_2\) at \(f_c(1)\) and/or \(f_c(w)\).

\(^2\)As a special case, one can verify that when \(\text{Im}(c) = 0, c = 1 - \alpha\) and \(\gamma = \frac{f_c(1) + f_c(w)}{2}\).
See Figure 2. When $c \in T_1$, $f_c(w)$ is undefined and $S_c$ is a vertical line through $f_c(1)$. Alternatively, one can construct $\alpha$, $f_c(1)$, and $\gamma$. Then $S_c$ is the circle with center $\gamma$ passing through $f_c(1)$. Unfortunately this method does not illustrate the intersection of $S_c$ and $T_2$.

We can also represent the corresponding polynomial analytically. As $r_1r_2$ is a chord of $S_c$ and $T_2$, $\overline{0\gamma}$ is a perpendicular bisector of $r_1r_2$. So there exists a $\theta$ with

$$r_1 = \frac{\gamma}{|\gamma|} e^{i\theta} \quad \text{and} \quad r_2 = \frac{\gamma}{|\gamma|} e^{-i\theta}.$$ 

See Figure 4.

Applying the Law of Cosines to $\triangle 0\gamma r_1$ gives

$$\cos(\theta) = \frac{1 + |\gamma|^2 - R^2}{2|\gamma|}.$$ 

For $l = \cos(\theta)$,

$$r_1, r_2 = \frac{\gamma}{|\gamma|} e^{\pm i\theta} = \frac{\gamma}{|\gamma|} \left(l \pm i \sqrt{1 - l^2}\right).$$
Property 2. Suppose \( c \in T_\alpha \) with \( \alpha \in (\frac{2}{3}, 1) \cup (1, 2) \) and \( c \neq -\frac{1}{3}, 1 \). Let

\[
\gamma = \frac{3c - 1}{2} + \frac{\alpha}{4\alpha - 4}, \quad l = \frac{|\gamma|^2 + 1 - \left(\frac{\alpha}{4\alpha - 4}\right)^2}{2|\gamma|}
\]

and

\[
r_1, r_2 = \frac{\gamma}{|\gamma|} \left( l \pm i\sqrt{1 - l^2} \right).
\]

Then \( p(z) = (z - 1)(z - r_1)(z - r_2) \) is the unique polynomial in \( \Gamma \) with critical point at \( c \).

For a simple example consider \( c \in T_2 \). Then \( c = e^{i\theta} \),

\[
\gamma = \frac{3e^{i\theta} - 1}{2} + \frac{2}{4} = \frac{3}{2} e^{i\theta} \text{ and } l = \frac{\frac{3}{2} + 1 - \frac{1}{3}}{2^2} = 1.
\]

Therefore, \( r = e^{i\theta} \left( 1 \pm i\sqrt{1 - 1^2} \right) = e^{i\theta} = c \) and

\[
p(z) = (z - c)^2(z - 1).
\]

This makes perfect sense, as \( c \in T_2 \) is the critical point of a \( p \in \Gamma \) if and only if \( p \) has a double root at \( c \).

References


