Final Report on the SAIF grant: Bounds on the Zeros of the Derivative of a Polynomial

The purpose of this grant was to provide support for the principal investigator to study the currently known methods for achieving bounds on the zeros of the derivative of a polynomial. The study of the zeros of a derivative begins in a first course in Calculus and continues on in virtually all areas of mathematical analysis.

The principal investigator found that a portion of this research was appropriate for a colloquium for our mathematics majors here at UW-Platteville. Indeed, a colloquium was given in fall 2008.

Also, a presentation of this research was given at the spring 2009 meeting of the Mathematical Association of America–Wisconsin Section.

Finally, a scholarly paper, submitted for publication in the research journal The American Mathematical Monthly, was written. A copy of this paper is given below.

Overall, the principal investigator deems this research project a success as all goals have been realized. It has been a pleasure to work on this project funded by a SAIF Grant.

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Bounds on the Zeros of the Derivative of a Polynomial

1 Introduction

Suppose a polynomial \( P(x) \) of degree \( n \geq 2 \) with real roots \( a_1 \leq \ldots \leq a_n \) is given. Rolle’s Theorem says that if \( a_k < a_{k+1} \), then the derivative \( P'(x) \) will have exactly one root \( b_k \) lying strictly between \( a_k \) and \( a_{k+1} \). We refer to \( b_k \) as a critical point of \( P(x) \). As a minimal example, suppose that the degree of \( P(x) \) is two. Write \( P(x) = (x-a_1)(x-a_2) \), where \( a_1 < a_2 \). Note that throughout this paper we will assume (without loss of generality) that the leading coefficient of our polynomial is one. Continuing, we have

\[
P(x) = x^2 - (a_1 + a_2)x + a_1a_2
\]
\[
P'(x) = 2x - (a_1 + a_2)
\]

so the only root of \( P'(x) \) is \( b_1 = \frac{1}{2}(a_1 + a_2) \). Notice that the root of \( P'(x) \) is the average of the two roots of \( P(x) \). In other words, \( b_1 \) lies halfway between \( a_1 \) and \( a_2 \). Since the degree two case is completely understood, in what follow we assume that the degree of the polynomial is greater than two.
2 Bounds on Critical Points

The following theorem due to Peyser [4] gives bounds on the critical points of a polynomial.

**Theorem 1** If a polynomial \( P(x) \) of degree \( n \geq 3 \) has only real roots \( a_1 \leq \ldots \leq a_n \) and if \( a_k < a_{k+1} \), then the unique root \( b_k \) of \( P'(x) \) between \( a_k \) and \( a_{k+1} \) satisfies the inequality

\[
a_k + \frac{1}{n-k+1} (a_{k+1} - a_k) \leq b_k \leq a_{k+1} - \frac{1}{k+1} (a_{k+1} - a_k).
\]

Melman’s [3] version of this theorem takes the multiplicity of the roots of \( P(x) \) into account.

**Theorem 2** If a polynomial \( P(x) \) of degree \( n \geq 3 \) has only real roots \( a_1 \leq \ldots \leq a_n \) and if \( a_k < a_{k+1} \), then the unique root \( b_k \) of \( P'(x) \) between \( a_k \) and \( a_{k+1} \) satisfies the inequality

\[
a_k + \frac{m_k}{n-k+m_k} (a_{k+1} - a_k) \leq b_k \leq a_{k+1} - \frac{m_{k+1}}{k+m_{k+1}} (a_{k+1} - a_k)
\]

where \( m_k \) is the multiplicity of the root \( a_k \).

Theorems 1 and 2 use the roots \( a_k \) and \( a_{k+1} \) to get bounds on \( b_k \). Here we show how to use all the roots \( a_1, \ldots, a_n \) to find bounds on \( b_k \).

**Theorem 3** If a polynomial \( P(x) \) of degree \( n \geq 3 \) has only real roots \( a_1 \leq \ldots \leq a_n \) and if \( a_k < a_{k+1} \), then the unique root \( b_k \) of \( P'(x) \) between \( a_k \) and \( a_{k+1} \) satisfies the following.

1. Define

\[
f_U(\beta) = (U_2 - U_1)(k) \beta^2 - \left[ 2 + (U_2 - U_1)(k) (a_k + a_{k+1}) \right] \beta
\]

\[
+ \left[ a_k + a_{k+1} + (U_2 - U_1)(k) a_k a_{k+1} \right]
\]

where

\[
U_1(k) = \sum_{j=1}^{k-1} \frac{1}{a_k - a_j}, \quad 2 \leq k \leq n-1, \quad U_1(1) = 0,
\]

and

\[
U_2(k) = \sum_{j=k+2}^{n} \frac{1}{a_j - a_k}, \quad 1 \leq k \leq n-2, \quad U_2(n-1) = 0.
\]

Then,

(a) \( b_k \) is bounded above by the smallest root of \( f_U(\beta) \) whenever \( (U_2 - U_1)(k) > 0 \),

(b) \( b_k \) is bounded above by the largest root of \( f_U(\beta) \) whenever \( (U_2 - U_1)(k) < 0 \), and

(c) \( b_k \) is bounded above by \( \frac{1}{2} (a_k + a_{k+1}) \) whenever \( (U_2 - U_1)(k) = 0 \).
2. Define

\[ f_L(\beta) = (L_2 - L_1)(k) \beta^2 - \left[ 2 + (L_2 - L_1)(k)(a_k + a_{k+1}) \right] \beta \\
+ \left[ a_k + a_{k+1} + (L_2 - L_1)(k) a_k a_{k+1} \right] \]

where

\[ L_1(k) = \sum_{j=1}^{k-1} \frac{1}{a_{k+1} - a_j}, \quad 2 \leq k \leq n-1, \quad L_1(1) = 0, \]

and

\[ L_2(k) = \sum_{j=k+2}^{n} \frac{1}{a_j - a_{k+1}}, \quad 1 \leq k \leq n-2, \quad L_2(n-1) = 0. \]

Then,

(a) \( b_k \) is bounded below by the largest root of \( f_L(\beta) \) whenever \( (L_2 - L_1)(k) > 0 \),
(b) \( b_k \) is bounded below by the smallest root of \( f_L(\beta) \) whenever \( (L_2 - L_1)(k) < 0 \),
and
(c) \( b_k \) is bounded below by \( \frac{1}{2}(a_k + a_{k+1}) \) whenever \( (L_2 - L_1)(k) = 0 \).

\textbf{Proof} \ If \( P(x) = (x-a_1)\cdots(x-a_n) \) and \( b_k \) is the unique root of \( P'(x) \) between \( a_k \) and \( a_{k+1} \), then \( P(b_k) \neq 0 \). Therefore,

\[ 0 = \frac{P'(b_k)}{P(b_k)} = \sum_{j=1}^{n} \frac{1}{b_k - a_j} \]

where the second equality follows by computing \( P'(x) \) using the product rule. Splitting the sum we have

\[ \sum_{j=1}^{k} \frac{1}{b_k - a_j} = \sum_{j=k+1}^{n} \frac{1}{a_j - b_k} \]

If \( j = 1, \ldots, k-1 \), then \( b_k - a_j \geq a_k - a_j \); therefore, for \( 2 \leq k \leq n-1 \)

\[ \sum_{j=1}^{k} \frac{1}{b_k - a_j} \leq \sum_{j=1}^{k-1} \frac{1}{a_k - a_j} + \frac{1}{b_k - a_k} = U_1(k) + \frac{1}{b_k - a_k}. \]

If \( k = 1 \), then the sum on the right side of the above inequality is nonexistent, so the inequality becomes an equality for \( k = 1 \).

Next, if \( j = k + 2, \ldots, n \), then \( a_j - b_k \leq a_j - a_k \); therefore, for \( 1 \leq k \leq n-2 \)

\[ \sum_{j=k+1}^{n} \frac{1}{a_j - b_k} \geq \frac{1}{a_{k+1} - b_k} + \sum_{j=k+2}^{n} \frac{1}{a_j - a_k} = \frac{1}{a_{k+1} - b_k} + U_2(k). \]

If \( k = n-1 \), then the sum on the right side of the above inequality is nonexistent, so the inequality becomes an equality for \( k = n-1 \).
So far we have shown
\[ U_1(k) + \frac{1}{b_k - a_k} \geq \frac{1}{a_{k+1} - b_k} + U_2(k). \]

for 1 \leq k \leq n - 1.

Replacing \( b_k \) with \( \beta \) we follow the algebra
\[
\frac{1}{\beta - a_k} - \frac{1}{a_{k+1} - \beta} - (U_2 - U_1)(k) \geq 0
\]
\[
(a_{k+1} - \beta) - (\beta - a_k) - (U_2 - U_1)(k) [(\beta - a_k)(a_{k+1} - \beta)] \geq 0
\]
\[
(a_{k+1} + a_k) - 2\beta - (U_2 - U_1)(k) [-\beta^2 + (a_{k+1} + a_k)\beta - ak_{a_{k+1}}] \geq 0
\]
to find that
\[
f_U(\beta) = (U_2 - U_1)(k) \beta^2 - \left[ 2 + (U_2 - U_1)(k)(a_k + a_{k+1}) \right] \beta
\]
\[
+ \left[ a_k + a_{k+1} + (U_2 - U_1)(k)a_ka_{k+1} \right] \geq 0.
\]

If \((U_2 - U_1)(k) > 0\), the graph of \( f_U(\beta) \) is a parabola that opens upward. Since we are solving \( f_U(\beta) \geq 0 \), where \( a_k < \beta < a_{k+1} \), it must be that \( b_k \) is bounded below by the largest root of \( f_U(\beta) \) or bounded above by the smallest root of \( f_U(\beta) \). Since,
\[
f_U(a_k) = a_{k+1} - a_k > 0 \quad \text{and} \quad f_U(a_{k+1}) = a_k - a_{k+1} < 0
\]
the largest root of \( f_U(\beta) \) is to the right of the interval \((a_k, a_{k+1})\) and \( b_k \) is bounded above by the smallest root of \( f_U(\beta) \).

A similar argument shows that \( b_k \) is bounded below by the largest root of \( f_U(\beta) \) whenever \((U_2 - U_1)(k) < 0\). Furthermore, if \((U_2 - U_1)(k) = 0\), then
\[
\frac{1}{b_k - a_k} \geq \frac{1}{a_{k+1} - b_k}
\]
\[
b_k \leq \frac{1}{2} (a_k + a_{k+1})
\]
as desired.

To prove the second part of the theorem we proceed in a similar manner. If \( j = 1, \ldots, k - 1 \), then \( b_k - a_j \leq a_{k+1} - a_j \); therefore, for \( 2 \leq k \leq n - 1 \)
\[
\sum_{j=1}^{k} \frac{1}{b_k - a_j} \geq \sum_{j=1}^{k-1} \frac{1}{a_{k+1} - a_j} + \frac{1}{b_k - a_k} = L_1(k) + \frac{1}{b_k - a_k}.
\]

If \( k = 1 \), then the sum on the right side of the above inequality is nonexistent, so the inequality becomes an equality for \( k = 1 \).

If \( j = k + 2, \ldots, n \), then \( a_j - b_k \geq a_j - a_{k+1} \); therefore, for \( 1 \leq k \leq n - 2 \)
\[
\sum_{j=k+1}^{n} \frac{1}{a_j - b_k} \leq \frac{1}{a_{k+1} - b_k} + \sum_{j=k+2}^{n} \frac{1}{a_j - a_{k+1}} = \frac{1}{a_{k+1} - b_k} + L_2(k).
\]
If $k = n - 1$, then the sum on the right side of the above inequality is nonexistent, so the inequality becomes an equality for $k = n - 1$.

Therefore,

$$L_1(k) + \frac{1}{b_k - a_k} \leq \frac{1}{a_{k+1} - b_k} + L_2(k).$$

for $1 \leq k \leq n - 1$.

Replacing $b_k$ with $\beta$ the same algebra as above results in

$$f_L(\beta) = (L_2 - L_1)(k) \beta^2 - \left[2 + (L_2 - L_1)(k) (a_k + a_{k+1})\right] \beta$$

$$+ \left[a_k + a_{k+1} + (L_2 - L_1)(k) a_k a_{k+1}\right] \leq 0$$

From here the fact that $b_k$ is bounded below by the appropriate root of $f_L(\beta)$ follows as above. Furthermore, if $(L_2 - L_1)(k) = 0$, then

$$\frac{1}{b_k - a_k} \leq \frac{1}{a_{k+1} - b_k},$$

$$b_k \geq \frac{1}{2} (a_k + a_{k+1}).$$

This finishes the proof.

**Note** If $(U_2 - U_1)(k) = (L_2 - L_1)(k) = 0$, then $b_k = \frac{1}{2} (a_k + a_{k+1})$.

**Example** Consider the polynomial $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$. In this case the estimates from Theorem 1 are

$$a_k + \frac{1}{6 - k} \leq b_k \leq a_{k+1} - \frac{1}{k + 1}$$

for $k = 1, 2, 3, 4$. In addition we apply Theorem 3 to find following results.

<table>
<thead>
<tr>
<th>Theorem 1</th>
<th>Theorem 3</th>
<th>Approximations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.20 \leq b_1 \leq 1.50$</td>
<td>$1.31 \leq b_1 \leq 1.37$</td>
<td>$b_1 \approx 1.36$</td>
</tr>
<tr>
<td>$2.25 \leq b_2 \leq 2.67$</td>
<td>$2.38 \leq b_2 \leq 2.52$</td>
<td>$b_2 \approx 2.46$</td>
</tr>
<tr>
<td>$3.33 \leq b_3 \leq 3.75$</td>
<td>$3.48 \leq b_3 \leq 3.62$</td>
<td>$b_3 \approx 3.54$</td>
</tr>
<tr>
<td>$4.50 \leq b_4 \leq 4.80$</td>
<td>$4.63 \leq b_4 \leq 4.69$</td>
<td>$b_4 \approx 4.64$</td>
</tr>
</tbody>
</table>

**Note** As Melman [3] took the multiplicity of the roots of $P(x)$ into account, so can we modify the proof of Theorem 3 to do the same. For example, at the beginning of the proof we write

$$\sum_{j=1}^{k} \frac{1}{b_k - a_j} \leq \sum_{j=1}^{k-m} \frac{1}{a_k - a_j} + \frac{m_k}{b_k - a_k}.$$
where $m_k$ is the multiplicity of the root $a_k$. Modifying the other inequalities in a similar fashion we arrive at upper and lower bounds for $b_k$.

**Example** Consider the polynomial $P(x) = (x - 1)(x - 2)^4(x - 3)^2(x - 4)$.

<table>
<thead>
<tr>
<th>Theorem 1</th>
<th>Theorem 2</th>
<th>Extension of Theorem 3</th>
<th>Approximations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.13 ≤ $b_1$ ≤ 1.50</td>
<td>1.13 ≤ $b_1$ ≤ 1.20</td>
<td>1.14 ≤ $b_1$ ≤ 1.17</td>
<td>$b_1$ ≈ 1.16</td>
</tr>
<tr>
<td>2.25 ≤ $b_5$ ≤ 2.83</td>
<td>2.57 ≤ $b_5$ ≤ 2.71</td>
<td>2.65 ≤ $b_5$ ≤ 2.68</td>
<td>$b_5$ ≈ 2.66</td>
</tr>
<tr>
<td>3.50 ≤ $b_7$ ≤ 3.88</td>
<td>3.67 ≤ $b_7$ ≤ 3.88</td>
<td>3.79 ≤ $b_7$ ≤ 3.85</td>
<td>$b_7$ ≈ 3.80</td>
</tr>
</tbody>
</table>

**References**


